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ANALYSIS OF THE LIGHT CURVES

OF

ECLIPSING VARIABLE STARS

by

© DAVID E. HOLMGREN

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled ANALYSIS OF THE LIGHT CURVES OF ECLIPSING VARIABLES submitted by David E. Holmgren in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

Four methods for determining the geometric elements of an eclipsing binary from its light curve are explored in detail. The methods discussed are those of Russell (specifically, the version due to Tabachnik), Kitamura, Kopal (frequency domain approach), and Wood. In each case, the underlying model of an eclipsing binary system is discussed. The various methods of light analysis are then applied to the eclipsing binaries HS Herculis, W Delphini, and HD 219634. The results of each analysis are discussed, and the various methods of analysis are compared with one another. Finally, the relative merits of each model of an eclipsing binary system are considered. Computer programs for the various methods of light curve analysis, along with explanations of their use, are presented in the appendices.

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CHAPTER 1

INTRODUCTION

Eclipsing binary stars are, in many ways, very informative to the astronomer and astrophysicist. The study of eclipsing binary stars can reveal a great deal about the sizes of stars, their masses, densities, and internal structure. Such information is also valuable in ascertaining the evolutionary state of the two (or more) stars constituting an eclipsing binary system. In certain cases in which the two component stars of an eclipsing binary system are in close proximity to one another, it is also possible to deduce the amount of tidal distortion present, and to check for the presence of matter streams between the two stars. With all of this information in hand, a scale model of an eclipsing binary star may be constructed, and hence, our knowledge of the system will be complete. One may then use this scale model to look for long-term effects such as apsidal motion, the presence of which can be deduced from a secular change in the period of the eclipsing binary. Such measurements may also be used to verify Einstein's theory of General Relativity.

The key problem, however, is in interpreting the observed light changes (the 'light curve') of an eclipsing binary. This is the critical step which lies between making the observations and coming to the conclusions outlined

in the previous paragraph. To this end a great deal of work has been done, from the first tentative steps taken by Russell in 1912 (Russell, 1912), which dealt with the determination of the geometric elements (the relative radii of the stars, the orbital inclination angle, and the relative luminosities) of an eclipsing binary consisting of non-limb darkened spherical stars, to the recent Fourier analysis methods of Kopal (see for instance, Kopal, 1979), which use the harmonic content of the observed light changes to deduce the geometric elements.

A necessary ingredient in all methods for determining the geometric elements from the observed light changes is a realistic physical model of the binary star, preferably involving as few assumptions as possible regarding the forms of the stars (i.e., spherical, non-spherical) and their physical properties (temperature, luminosity, etc.). The complexity, and consequently the realism, of such models of eclipsing binaries has grown since the initial investigation of the problem by Russell in 1912. Present day models of eclipsing binaries describe a range of situations, from that of two well-separated spherical stars to systems in which both stars are in contact, in which case both stars are greatly distorted by their mutual tidal interaction. A question of some importance is then: which of the several models of eclipsing binary stars currently available best describes a given eclipsing binary star? To answer this question, representative models of eclipsing binary stars

and their accompanying methods of light curve analysis will have to be analyzed and the appropriate conclusions drawn.

There are three broad categories of methods used in light curve analysis. They are the "classical" or "Russell-type" methods previously referred to, the "synthesis" methods, devised in the early 1970s, and the "frequency-domain" methods of Kopal previously referred to. Two methods of the "classical" type are the Russell-Merrill (1952) method and the method of Kitamura. A fine example of a "synthesis" method is a method devised by Wood. Several versions of Kopal's frequency-domain method exist, but the best of these are the most recent ones (e.g., Kopal (1982)). These methods of light curve analysis will be applied to three stars covering a wide range of physical conditions, from the well-separated case of W Delphini, to the case of HS Herculis with its matter stream, and finally to HD 219634, which may be a massive binary and possibly even an X-ray source (see Gulliver, Hube and Lowe (1982)). This analysis will, we hope, shed some light on the validity of the various models of eclipsing binary stars to be considered.

CHAPTER 2

THE RUSSELL MODEL

2.1 General Principles

The first steps toward an interpretation of eclipsing binary light curves were taken by Russell in 1912 (Russell (1912a,b)). Subsequent refinements to the theory were made by Russell and two collaborators, Merrill and Shapley (see Russell and Shapley (1912); Russell and Merrill (1952)). The Russell model can be applied to both spherical and non-spherical stars with varying degrees of accuracy.

The "spherical model" assumes that the eclipsing binary system consists of two spherical stars moving in circular orbits about a common centre of gravity. The distribution of surface brightness $J(\gamma)$ over the disk of each star is assumed to follow the "cosine law"

$$J(\gamma) = J(0)(1 - x + x\cos\gamma) \tag{2.1}$$

where $J(0)$ is the surface brightness at the centre of the observed disk of either star, x the coefficient of limb darkening, and γ the angle of foreshortening, or the angle between the line of sight and a radius vector from the centre of the star (see figure 1). The angle γ varies between 0 and 90 degrees. At this point, it should be noted that the physical properties of the stars enter the Russell model only through equation (2.1). The detailed features of the stars (e.g., starspots, magnetic fields,

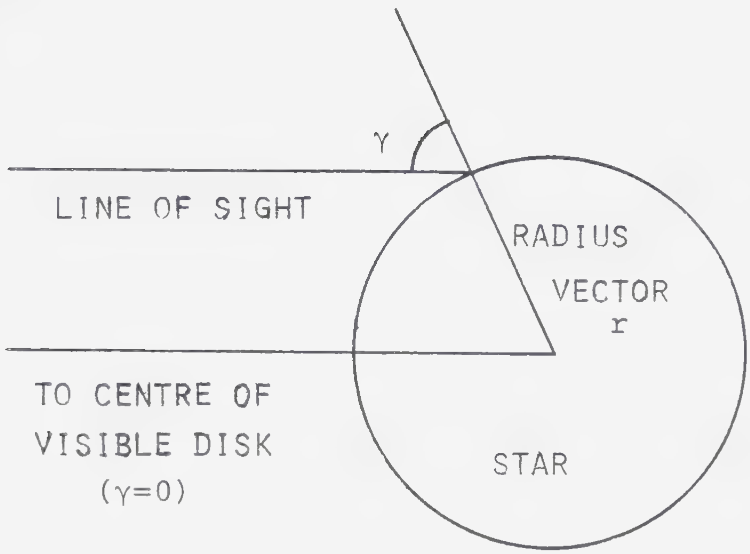


FIGURE 1. DEFINITION OF γ, THE ANGLE OF FORESHORTENING.

intrinsic variability) are not accounted for.

To arrive at a method for determining the geometric elements of an eclipsing binary using the spherical model, it will be necessary to consider, in some detail, the geometry of an eclipse. The treatment that follows is similar to that given by Irwin (1962) or Russell (1912a,b). Before plunging headlong into this problem, it will be necessary to define some of the quantities that will be used. The radius of the smaller star is denoted by r_s, and that of the larger star by r_g. These radii are measured in units of the centre-to-centre separation of the component stars of an eclipsing binary. The relative luminosity of the small and large stars will be denoted by L_s and L_g, respectively. These luminosities are so defined that

$$L_s + L_g = 1 \ . \qquad (2.2)$$

Since an eclipsing binary light curve displays brightness as a function of time, it is necessary to define an orbital phase θ by

$$\theta = \frac{2\pi}{P}(t - t_o)$$

where P is the period of the eclipsing binary, t the time at which the brightness was observed or is to be calculated, and t_o the time of conjunction, which usually coincides with the time of minimum light during the primary (deeper) eclipse. The times t and t_o are measured in Julian days, while the period P is measured in days. With explicit

reference to the eclipse, it is customary to define three additional quantities, namely k, p, and δ. The dimensionless parameter k is simply the ratio of the radii r_s and r_g:

$$k = \frac{r_s}{r_g} \quad , \qquad (0 \leq k \leq 1) \ .$$

The "geometric depth" p is another dimensionless parameter, which represents the extent to which the eclipse has progressed at any eclipse phase θ. A parameter closely related to p is δ, the apparent separation of the centres of the disks of the stars. The parameters p and δ are shown in figure 2. The following equation gives the relationship between p and δ (and does, in fact, serve to define p)

$$p = \frac{\delta - r_g}{r_s} \tag{2.3a}$$

or

$$\delta = r_g(1 + kp) \quad . \tag{2.3b}$$

using the definition of k stated above. Relation (2.3b) is the more useful of the two relations relating p and δ. The quantity δ may also be related to the phase θ and the orbital inclination. The orbital inclination is defined as the angle between a plane perpendicular to the line of sight (the "celestial sphere") and the orbital plane (see figure 3). Through the use of some simple trigonometry, and recalling that the two stars constituting the eclipsing binary have a unit separation, one arrives at the "geometrical relation":

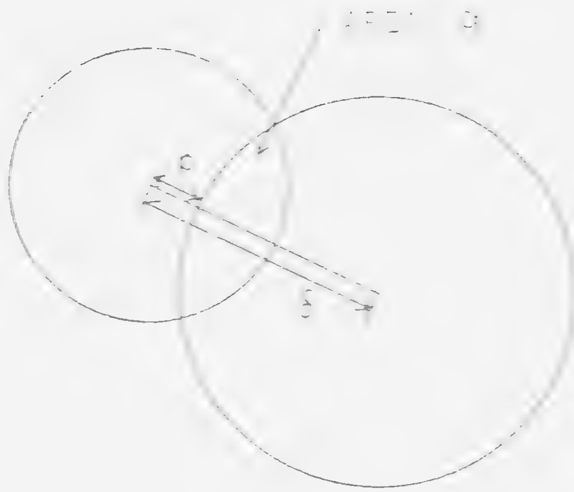


FIGURE 2. [illegible]
[illegible]

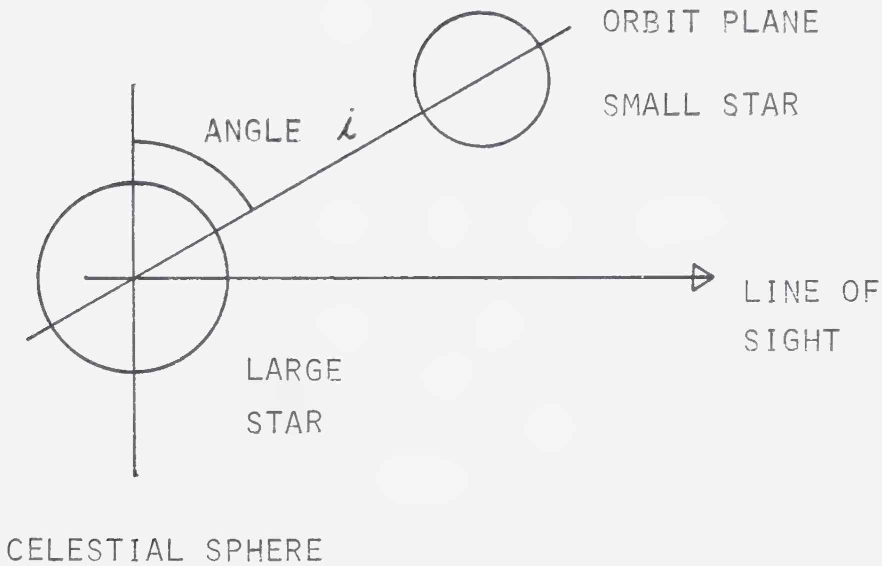


FIGURE 3. DEFINITION OF THE ANGLE i.

$$\delta^2 = \cos^2 i + \sin^2\theta \sin^2 i = 1 - \sin^2 i \cos^2\theta$$

or

$$r_g^2(1 + kp)^2 = \cos^2 i + \sin^2\theta \sin^2 i \quad . \tag{2.4}$$

Equation (2.4) is the fundamental equation of all Russell-type methods for the determination of the geometric elements. The derivation of the geometric relation is outlined in figure 4. It should also be noted that only a relative orbit is considered, namely, the relative orbit of the smaller star about the larger one. In the case of an elliptical orbit, the geometric relation would be multiplied by R^2, R being the separation between the stars at any orbital phase. The orbital phase would have to be replaced by $v-\omega$, where v is the true anomaly and ω the angle between the line of apsides and the line of sight. The vast majority of eclipsing binary systems have circular orbits, however, largely as a consequence of their short orbital periods and consequent tidal interations.

As an aid in the interpretation of eclipsing binary light curves, a relative luminosity ℓ is defined. This luminosity is related to a change in magnitude Δm by

$$\ell = 10^{-0.4\,\Delta m} \tag{2.5}$$

where Δm is taken relative to the magnitude of the eclipsing binary just before the start of the eclipse. The value of ℓ at the minimum of the eclipse is denoted by λ. A quantity $\alpha = \alpha(k,p)$ is also defined; representing the fractional

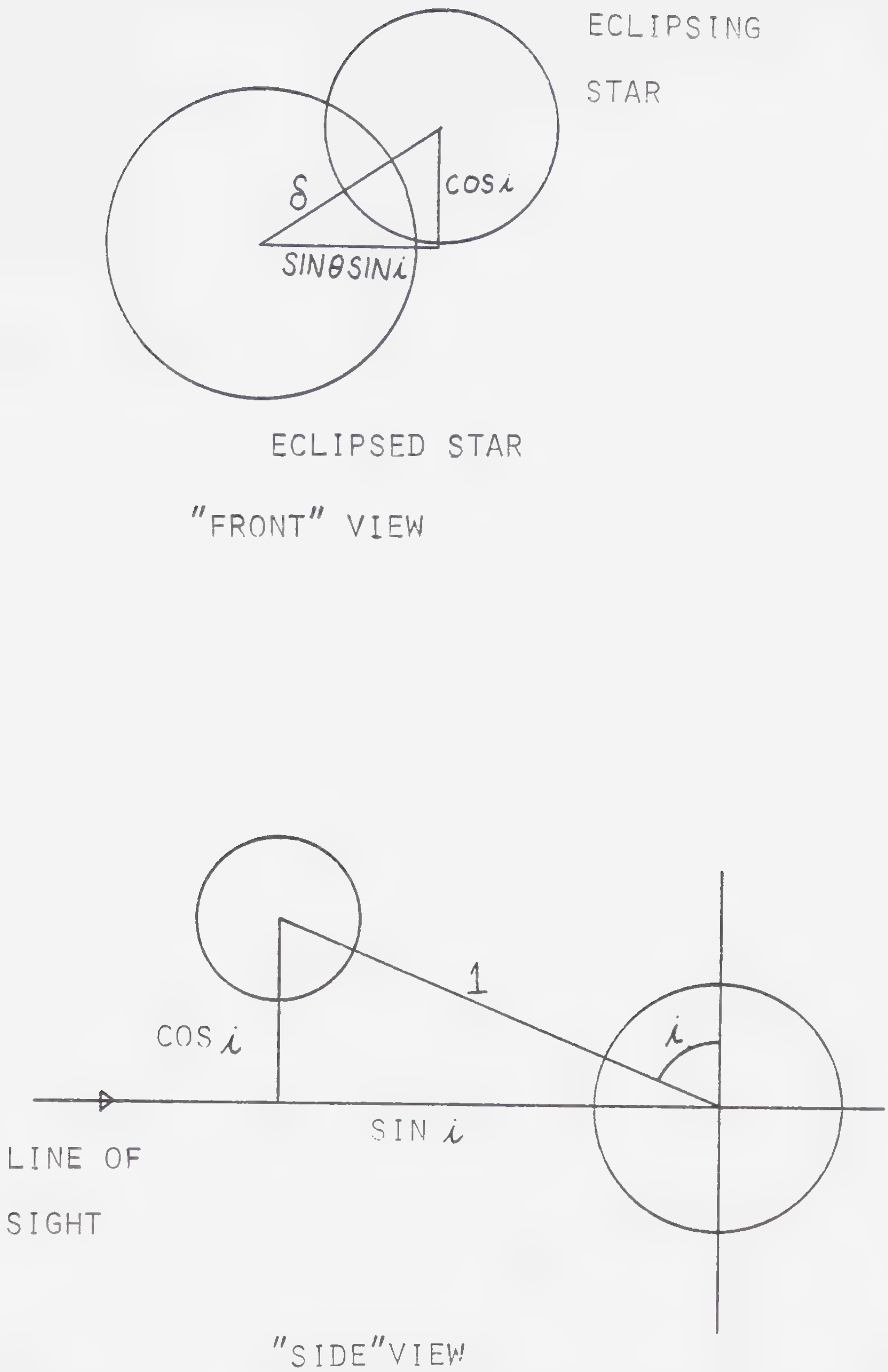


FIGURE 4. THE GEOMETRIC RELATION.

light loss. At any eclipse phase (see figure 2). This quantity may be determined directly from the observations by

$$\alpha = \frac{1 - \ell}{1 - \lambda} \qquad (2.6)$$

$$\ell = 1 - \alpha L \quad .$$

2.2 Total and Annular Eclipses

We are now in a position to describe Russell's method for the determination of r_s, r_g, and i. The following derivation may also be found in a recent book by Kopal (1979, pg. 110). The basic idea of the method is to write down the geometric relation (eqn. (2.4)) for three eclipse phases and to consider $\sin^2 i$, $\cos^2 i$, and r_g^2 as the unknowns. For the system of equations to have a unique solution,

$$\begin{vmatrix} \sin^2\theta_1 & (1+kp_1)^2 & 1 \\ \sin^2\theta_2 & (1+kp_2)^2 & 1 \\ \sin^2\theta_3 & (1+kp_3)^2 & 1 \end{vmatrix} = 0 \quad . \qquad (2.7)$$

In this equation, k is the only unknown, since p can in principle be determined from $\alpha(k,p)$. The phases θ_1 and θ_2 are chosen so as to correspond to $\alpha = 0.6$ and $\alpha = 0.9$ respectively. From this point on, the method used to determine r_s, r_g and i depends on the type of eclipse. It should also be noted that the entire light curve is not required for the analysis. Only one half of an eclipse is required.

For a total or annular eclipse (see figure 5), the determinant in equation (2.7), is written as

$$\sin^2\theta_3 = A + B\,\psi(k,p) \tag{2.8}$$

where

$$A = \sin^2\theta_1 \quad , \quad B = A - \sin^2\theta_2$$

and

$$\psi(k,p,\alpha) = \frac{2(p_3 - p_1) + k(p_3^2 - p_1^2)}{2(p_1 - p_2) + k(p_1^2 - p_2^2)} \quad . \tag{2.9}$$

If $\sin^2\theta_3$ is allowed to represent any eclipse phase, then A and B may be determined, and finally $\psi(k,p,\alpha)$ for the given θ_3. Thus, one tabulates $\psi(k,p,\alpha)$ for all eclipse phases. $\psi(k,p,\alpha)$ can also be tabulated using equation (2.9), so a comparison between the observed and theoretical values of $\psi(k,p,\alpha)$ can be made, allowing k to be determined for each eclipse phase. Russell tabulated $\psi(k,\alpha)$ for both types of eclipse, but the most comprehensive tabulation was that of Russell and Merrill (1952). A shorter and more useful set of tables (for $x = 0.5$) was published by Irwin (1962). To summarize, one finds a value for k by determining $\psi(k,\alpha)$ from the light curve by equation (2.8), and by using these observed values of $\psi(k,\alpha)$, along with the corresponding values of α determined by equation (2.6), to do inverse interpolation in a table of $\psi(k,\alpha)$, thereby producing a range of values for k. An average of the values of k is taken, and this number, $\langle k \rangle$, is then taken to be the 'correct k' in later calculations. The inclination i and the radius r_g can now be found from

$$\cot^2 i = \frac{B}{\phi_2(k)} - A \quad \text{and} \quad (r_g \csc i)^2 = \frac{B}{\phi_1(k)} \,, \qquad (2.10)$$

where $\phi_1(k)$ and $\phi_2(k)$ are two auxiliary functions also tabulated by Russell and Merrill in the reference quoted above. The value of r_s can now be found by using the definition of k.

The method of finding r_s, r_g, and i just described was modified by Russell and Merrill (1952) to use more points on the light curve during an eclipse. This is achieved by using three weighted means of $\sin^2\theta$ and $\psi(k,\alpha)$, and by defining a new function $R(x,k)$

$$R(x,k) = \frac{M_1[\sin^2\theta] - M_2[\sin^2\theta]}{M_2[\sin^2\theta] - M_3[\sin^2\theta]} = \frac{M_1[\psi] - M_2[\psi]}{M_2[\psi] - M_3[\psi]} \,, \qquad (2.11)$$

$$M_j[\sin^2\theta] = A + B\, M_j[\psi] \,, \quad j = 1,2,3$$

where $M_j[\sin^2\theta]$, $j = 1,2,3$, is a weighted mean of $\sin^2\theta$ for certain values of α, and $M_j[\psi]$, $j = 1,2,3$, is the corresponding weighted mean of $\psi(k,\alpha)$. Only one table is required to find k given $R(x,k)$, $M_1[\sin^2\theta]$, $M_2[\sin^2\theta]$, and $M_3[\sin^2\theta]$. The values of r_s, r_g and i are obtained as in the earlier version of the Russell method. The version just described, known as the 'Russell-Merrill' method, has the advantage of simplicity and greater computational speed, and will be used in subsequent examples.

2.3 Partial Eclipses

In the case of a partial eclipse (see figure 5), a different approach is required. The problem is more difficult to solve since observations of both eclipses are required for a unique solution, and the value of α at mid-eclipse (denoted by α_o) is also unknown. At mid-eclipse, the geometric relation (2.4) becomes

$$\cos^2 i = r_g^2 (1 + kp_o)^2 \quad , \quad p_o = p(k,\alpha_o) \tag{2.12}$$

since $\theta = 0$ at this point. Upon subtracting this result from the geometric relation, one has

$$\sin^2\theta = (r_s r_g \csc^2 i)[2(p - p_o) + k(p^2 - p_o^2)] \ . \tag{2.13}$$

This is the fundamental equation for the analysis of partial eclipses. Russell's approach (Russell, 1912b) was to first define n as the ratio of $1-\ell$ to $1-\lambda$ at any eclipse phase, and to take the value of $\sin^2\theta$ at $n = 0.5$ as a 'base point'. Russell then divided equation (2.13) by its counterpart at $n = 0.5$, obtaining

$$\frac{\sin^2\theta(n)}{\sin^2\theta(0.5)} = \frac{2(p - p_o) + k(p^2 - p_o^2)}{2(p_1 - p_o) + k(p_1^2 - p_o^2)} \equiv \chi(k,\alpha_o;n) \tag{2.14}$$

where $\theta(0.5)$ denotes the value of θ when $n = 0.5$, $\theta(n)$ the value of θ for any other n, and p_1 the value of p at $n = 0.5$. In the analysis of partial eclipses, the χ-functions play a similar role as do the ψ-functions in the analysis of total and annular eclipses. However, in the partial eclipse case,

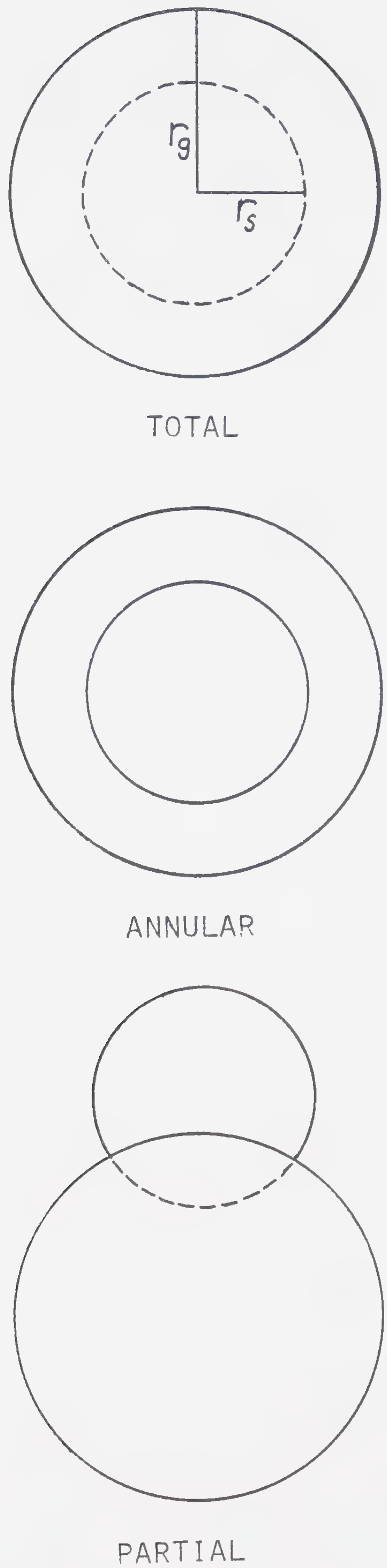


FIGURE 5. ECLIPSE TYPES.

the solution is graphical. The values of $\chi(k,\alpha_o;n)$ may be computed from the light curve by using the left-hand side of equation (2.14). Since k and α_o are to be solved for first, only two values of χ are needed. Suppose these values to be denoted by $\chi = c_1$ and $\chi = c_2$. One may also compute χ from the right-hand side of equation (2.14), and therefore tabulate $\chi(k,\alpha_o;n)$. The most complete tables of $\chi(k,\alpha_o;n)$ are those compiled by Russell and Merrill (1952), which can be used for both occultation and transit eclipses and any value of limb darkening x. After choosing a value of x, one uses the χ-tables to plot α_o as a function of k for the given values of χ, namely c_1 and c_2. Therefore, the point at which these curves intersect should provide the required values of k and α_o. Unfortunately, the solution obtained is indeterminate since it is not known whether the given eclipse is an occultation or a transit. Therefore, both eclipses must be used. The type of eclipse may now be determined quite easily by using the relationship (see Irwin (1962), p. 607)

$$\chi^{oc}(k,\alpha_o;n=0.8) > \chi^{tr}(k,\alpha_o;n=0.8) \qquad (2.15)$$

where 'oc' denotes occultation and 'tr' transit. This relation may be verified by consulting the appropriate tables of χ for $n = 0.8$. Another problem arises in the fact that the intersection of the two χ = constant curves can be quite shallow, resulting in an indeterminate solution once again. To remove this indeterminacy, another independent

relation must be introduced.

If λ denotes the value of minimum light for either eclipse, then

$$\lambda = 1 - \alpha_o L \qquad (L_s + L_g = 1)$$

where L is the relative luminosity of either star. If one writesthis out for both eclipses and solves for α_o, there results

$$\alpha_o^{oc} = 1 - \lambda_a + \frac{1 - \lambda_b}{k^2 Y} \qquad \text{for an occultation}$$

$$(2.16)$$

$$\alpha_o^{tr} = 1 - \lambda_b + (1 - \lambda_a)k^2 Y \qquad \text{for a transit}$$

where λ_a and λ_b represent λ for occultation and transit eclipses respectively and Y denotes the ratio $\alpha_o^{oc}/\alpha_o^{tr}$. Either of equations (2.16) is known as a "depth" equation, since such equations relate the depth of an eclipse $(1-\lambda)$ to α_o and k. Equations (2.16) are incorporated in the solution method for partial eclipses by obtaining values of k and α_o for successive values of Y. Tables of $Y(\alpha_o, k)$ exist (Irwin (1962) gives tables of $q_o(k,\alpha_o^{oc}) = k^2 Y(k,\alpha_o^{oc})$) for this purpose. The set of values of k and α_o so obtained are plotted on the same graph as are the equations for χ = constant mentioned earlier. The curve so defined will usually make a steep intersection with the χ = constant curves, thereby rendering the solution determinate. An example of such a graph is shown in figure 6.

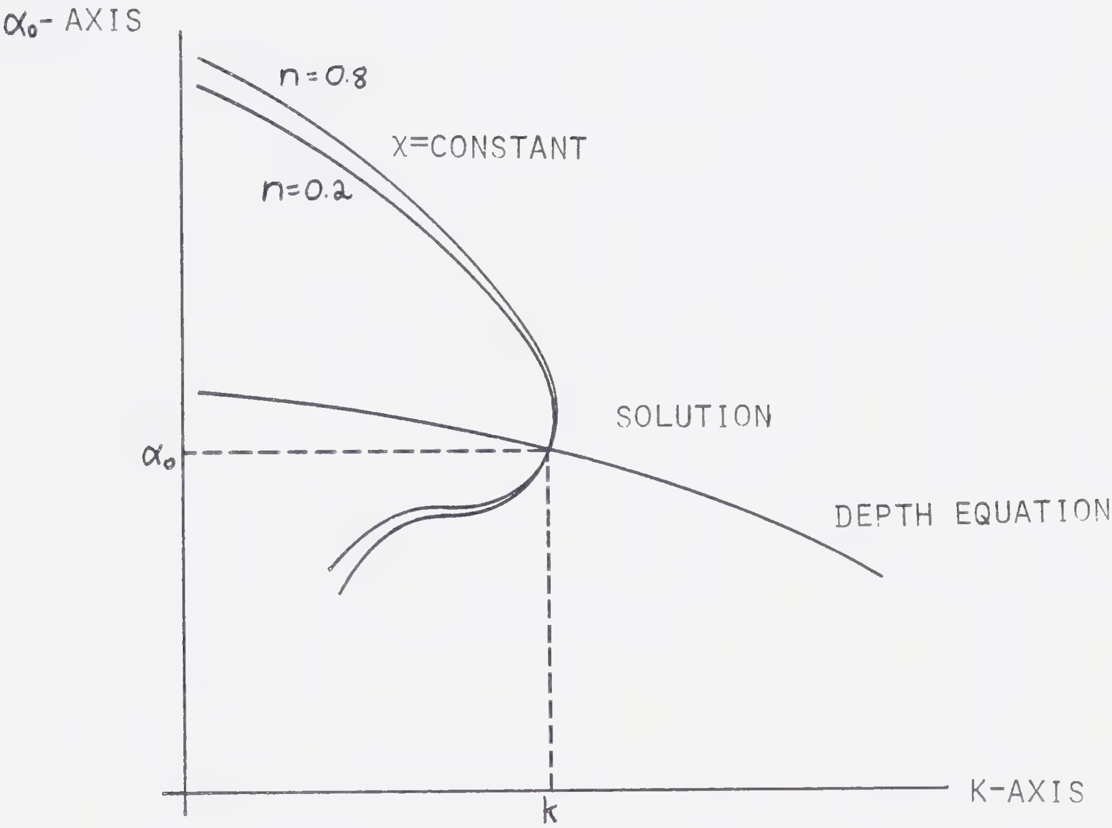


FIGURE 6. THE GRAPHICAL SOLUTION FOR PARTIAL ECLIPSES (ADAPTED FROM IRWIN(1962,PG.607)).

The preceding paragraphs have described the form of the Russell-Merrill method that is most useful for the analysis of total and annular eclipses, but which for partial eclipses is not the best nor the most useful approach. In fact, Kopal (Kopal, 1979, pp. 113-114) has argued strongly against the use of the χ-functions for determining the orbital elements. The essence of his criticism is that the position of any 'fixed' or 'base' points, as used in the Russell-Merrill method, can be determined only to some finite accuracy, and that this uncertainty would propagate through the entire light curve solution, leading to an uncertainty in the values of r_1, r_2 and i. Furthermore, the solution is fitted only at the 'fixed points', not at all of the data points. One could imagine a "worst case" in which a rather large initial error would propagate and magnify through the solution, leading to wildly erroneousresults. It is situations such as these which have led other workers to use other versions of the Russell-Merrill method.

2.4 A General Formulation

The Russell-Merrill method may be restated in a form useful for any type of eclipse, and moreover, in a form that is amenable to use with electronic computers. This method, due to Kopal (see Kopal (1979), pg. 115), takes the geometric relation and rewrites it in the form $y = ax + b$, which is linear. If one defines

$$x = \sin^2\theta \quad \text{and} \quad y = (1 + kp)^2$$

then the geometric relation may be written in the form

$$x = (r_g \csc i)^2 y - \cot^2 i$$

or

$$y = \frac{\sin^2 i}{r_g^2} x + \frac{\cos^2 i}{r_g^2} \quad , \tag{2.17}$$

this latter form being suggested by Tabachnik (1973). The elements r_g, r_s, and i may be found from the following equations, where $a = \sin^2 i / r_g^2$ and $b = \cos^2 i / r_g^2$:

$$\tan^2 i = \frac{a}{b} \quad , \quad r_g = (a+b)^{-1/2} \quad \text{and} \quad r_s = kr_g \quad . \tag{2.18}$$

If an initial value of k is used to determine p from a table of $\alpha(k,p)$, then the correct value of k will be the one that renders equation (2.17) a straight line. The straight line is fitted using the standard least-squares techniques. A good initial guess at k can be made in several ways. The simplest is to use the formula

$$k = \frac{\theta' - \theta''}{\theta' + \theta''}$$

where θ' is the phase angle of first contact and θ'' the phase angle of second contact (see figure 7 for definitions of θ' and θ''). Other methods are given in Appendix 1. The advantage in using equation (2.17) lies in the fact that all available eclipse data are used, and no special points on the light curve are required. Moreover, one can use any $\alpha(k,p)$ table, for either an occultation or a transit, and for any limb darkening to determine $p(k,\alpha)$. Therefore, this version of the Russell method is clearly the preferable one.

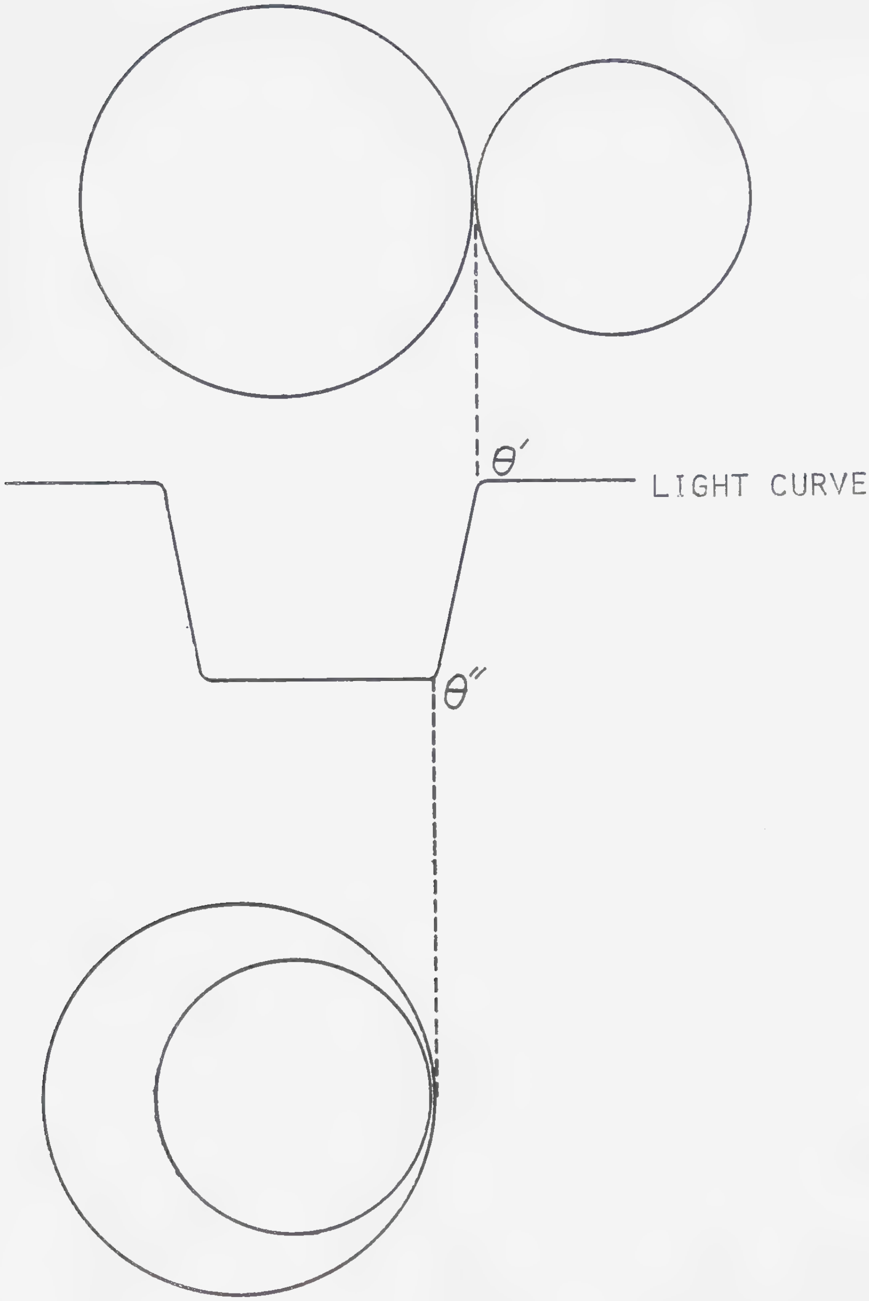


FIGURE 7. THE PHASES OF FIRST AND SECOND CONTACT (θ´,θ´´).

The present author has written a program for the Tl-59 programmable calculator to use equation (2.17) in the analysis of eclipsing binary light curves. The values of $\sin^2\theta$ and p are used as input. The Tl-59 calculator is particularly convenient since it has a built-in least-squares linear fit routine, which can be easily incorporated into a larger program. This program will be used in later sections when particular stars are considered. A listing of the program is presented in Appendix 1. A computer program, LINE, incorporating Tabachnik's method, is also listed in Appendix 1.

Before discussing the application of the Russell method and the Russell-Merrill method to close eclipsing binary stars, it should be mentioned that several other versions of the Russell-Merrill method exist in the literature. Most of these are due to Kopal, in particular the iterative methods (based on equation (2.13)), which have proven to be very useful. These methods are conveniently summarized in the 1979 book by Kopal. A computer program incorporating an iterative method has been published by Jurkevich (1970). An important variation due to Kitamura (1965), which employs Fourier transforms of the light curve, will be considered in the next section. Some methods which are no longer in use are those of Scharbe (1925) and Fetlaar (1923). The latter method is summarized in a book by Tsesevich (1973), which also contains a description of a method called the 'express method'. A recent revival of Kopal's iterative methods may be found in Look et al. (1978),

which also contains an interesting application of the depth equation.

2.5 Non-Sphericity and Rectification

Naturally, one cannot apply the Russell-Merrill method (or any one version of it) to all eclipsing binary stars. Not all eclipsing binary stars have spherical component stars since there are inevitably tidal effects in any close system. Those eclipsing binaries with relatively short periods, less than about 3 days, will most certainly have some tidal distortion present, since the two stars involved will be quite close to one another (Kepler's harmonic law: $P^2 \propto a^3$). There are often other associated effects. An obvious one is that one star will heat the other, the effect being a mutual one. When first discovered, this effect was called the "reflection effect", since it was believed at the time that light from one star was reflecting off the surface of the other: Though inaccurate, the name stuck. As the theory of stellar atmospheres evolved beyond the well-known "gray" case, it was realized that the "reflection" effect was really a heating effect. The reflection effect has become one of the most difficult effects to understand, and hence model, since the magnitude of the effect depends not only on the closeness of the stars, but also on the state of their atmospheres. The problem as it currently stands is summarized by Sahade and Wood (1978). A comprehensive study of the reflection effect, typical of many done, is that done by Napier (1968).

Another effect present in eclipsing binaries is a direct consequence of the closeness of the component stars. This is the presence of streams of matter between the stars. Matter streams can arise in two ways, the first being the presence of one star with a moving atmosphere (stellar wind). Wolf-Rayet stars and red giant stars can be involved in this type of mass exchange. A mass exchange can also arise if one star expands out to its Roche limit (see figure 8). Some of the matter from the expanding star is then drawn off by the other star (through its gravitation), with the result being either an accretion disk or a "hot spot", where the matter stream makes contact with the atmosphere of the attracting star. In the Russell model, the effects of tidal interaction, reflection, and mass transfer are dealt with by the process of rectification.

The dynamical and physical theory upon which the process of rectification rests will not be developed here. A comprehensive treatment may be found in an article by Martynov (1973) in the book edited by Tsesevich (1973). The treatment to be followed here is that given by Proctor and Linnell (1972). The Russell model treats the stars of a close eclipsing binary as prolate spheroids (see figure 9), although the results obtained at the end of the rectification process can be converted into results applying to a triaxial ellipsoid. The object of rectification is to convert the light curve of an eclipsing binary consisting of distorted stars into an equivalent "spherical" light

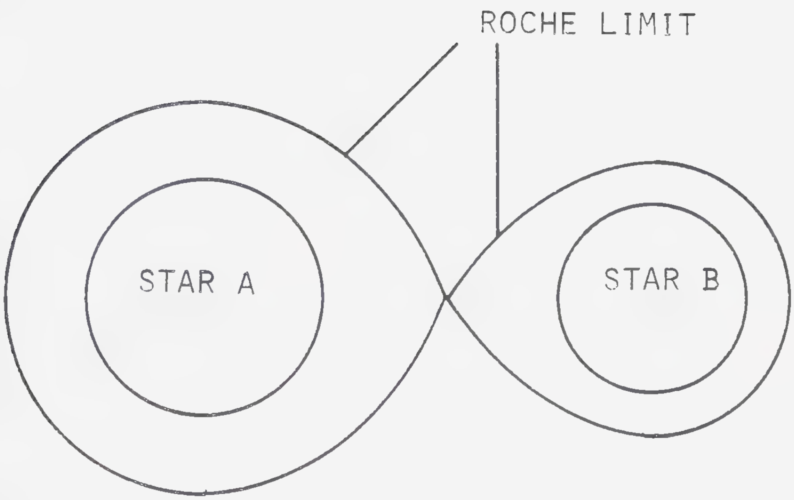


FIGURE 8. THE ROCHE SURFACE.

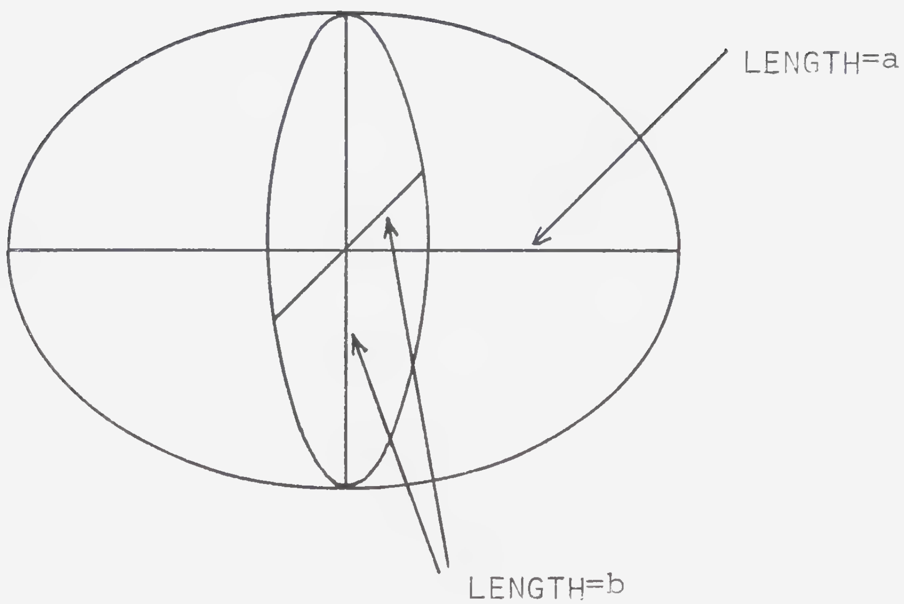


FIGURE 9. A PROLATE SPHEROID.

curve. Thus, rectification produces a rectified luminosity ℓ_r and a rectified phase θ_r, given unrectified values of ℓ and θ. More rigorously, if an observer at point 0, having direction cosines ℓ, m, n is watching an ellipsoidal star in a close binary system, then the process of rectification is an affine transformation that carries the observer at 0 to another point 0', with direction cosines ℓ', m', n', at the same distance from the sphere. Since the transformation carries an ellipsoid into a sphere (with a radius equal to the ellipsoid's semi-major axis), then the luminosity of the spherical star, as seen at 0', must be the same as that seen from the ellipsoid at 0. Since the light from the ellipsoid varies with phase, the light from the sphere must be modulated to produce the same light variation. The affine transformation is illustrated in figure 10. It was shown by Russell and Merrill (1952) that the light variation from a prolate spheroid with axes (a,b,b) is the same as that from a triaxial ellipsoid having axes (a,b,c). If the orbit has an inclination j in the prolate spheroid case and i in the ellipsoid case, then the semiaxes b and c of the triaxial ellipsoid are related by

$$\frac{\tan^2 j}{\tan^2 i} = \frac{c^2}{b^2} \quad . \tag{2.19}$$

Before proceeding further, it is necessary to define some parameters that will be used in the discussion that follows. The oblateness ε is defined as (a-b)/a, a and b being the semiaxes of either the prolate spheroid or the

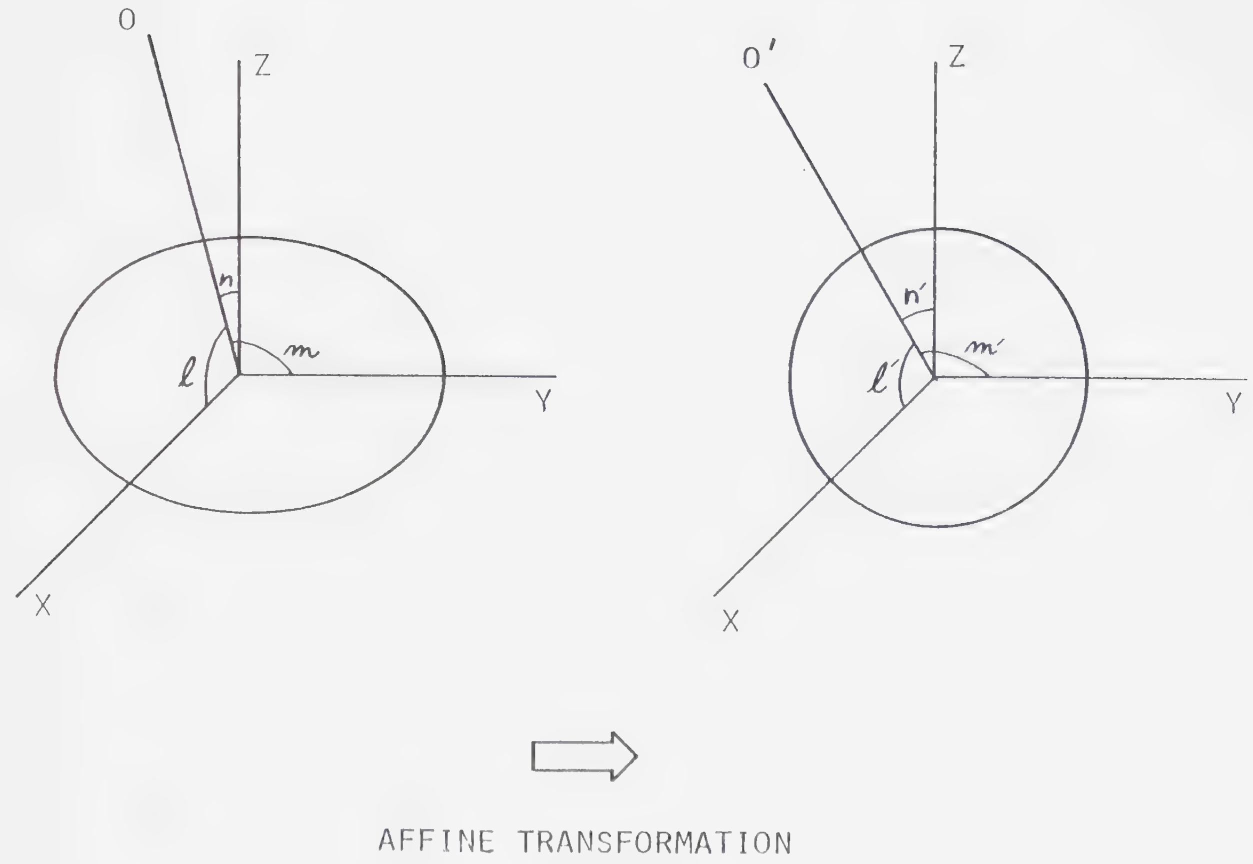


FIGURE 10. THE RECTIFICATION PROCESS.

triaxial ellipsoid. An approximate expression for ε is $\frac{1}{2}\eta^2$, η being the eccentricity of the cross-section of the star in the orbital plane. It will also be useful to define z, which is equal to $2\varepsilon \sin^2 j$. A non-spherical star will not have a uniform surface gravity, and consequently, those parts of the surface of the star farther from the star's center will appear cooler, while those parts closer to the center (near the pole of rotation) will appear hotter. A quantity that describes this effect is the gravity darkening coefficient y, which is defined by Martynov (1973):

$$y = \frac{c_2}{4\lambda T_o(1 - e^{c_2/\lambda T_o})} \tag{2.20}$$

where λ is the wavelength of observation, T_o the surface temperature, and c_2 (equal to hc/k) is a numerical constant whose value depends on the units of λ and T_o (see Gray (1976), pg. 117). Therefore, the observed intensity at any point on the star's surface will be

$$I = H(1 - x + x \cos\gamma)(1 - y + y g/g_o) \tag{2.21}$$

where H is the intensity at the centre of the observed disk, x and γ are defined in equation (2.1), g is the surface gravity at any point on the star, and g_o is a reference value of g, usually taken to be the value at the pole of the star. The light from either star can be expressed as (Russell and Merrill (1952), pg. 42):

$$\ell(\theta) = I(90^\circ)(1 - N\varepsilon \sin^2 j \cos^2\theta) + G f(\phi) \qquad (2.22)$$

where

$$N = \frac{15 + x}{15 - 5x}(1 + y) \; .$$

In this formula, $I(90^\circ)$ is the light from one star at quadrature phase ($\theta = 90^\circ$), G is an 'albedo' factor that determines the fraction of light received from the companion star which is reradiated at the wavelength of observation (Russell and Merrill (1952), pg. 46), and $f(\phi)$ is a 'phase function' that characterizes the reflection effect. The procedure of rectification is one in which the effects of reflection and ellipticity are removed by writing out the equation for $\ell(\theta)$ for both stars, finding the sum of these two equations, and then doing the appropriate subtraction and division to produce the value of $\ell(\theta)$ for a system consisting of spherical stars. In practice, rectification is done by fitting a Fourier series of the form:

$$\ell(\theta) = A_o + A_1 \cos\theta + A_2 \cos 2\theta + B_1 \sin\theta + B_2 \sin 2\theta \qquad (2.23)$$

to the light curve of the eclipsing binary outside the eclipses (one may use $\cos^2\theta$ and $\sin^2\theta$ instead of $\cos 2\theta$ and $\sin 2\theta$, by making use of a trigonometric identity, but the coefficients will then take on different meanings). The series may be fitted either by the least-squares method, or by a graphical method developed by Russell and Merrill. An example of the latter may be found in Appendix 1. The rectified light is then given by

$$\ell_{rect} = \frac{\ell_{obs} - (B_1 \sin\theta + B_2 \sin 2\theta)}{\frac{1}{2} A_o + A_1 \cos\theta + A_2 \cos 2\theta} \qquad (2.25)$$

and the rectified phase by

$$\sin\theta_{rect} = \frac{\sin\theta_{obs}}{(1 - z\cos^2\theta_{obs})^{\frac{1}{2}}}$$

and

$$\cos\theta_{rect} = \left(\frac{1 - z}{1 - z\cos^2\theta_{obs}}\right)^{\frac{1}{2}} \cos\theta_{obs} \quad , \qquad (2.26)$$

where both equations are required for proper quadrant definition (in taking an inverse tangent, an electronic computer uses the range $-\pi/2 \leq \theta \leq \pi/2$, rather than $0 \leq \theta \leq 2\pi$, which is the range of θ_{obs}). These formulae apply to all observations, both in and out of eclipse. The factor z may be obtained empirically by using the approximate relation $z \simeq |2A_2|$. It should also be noted that the presence of the sine terms in the Fourier series for $\ell(\theta)$ is not justifiable physically; their only purpose is to take care of any extra 'complications' that might arise. This then is the process of rectification as developed by Russell and Merrill. The end product is a light curve that is flat outside the eclipses, with the eclipses being those appropriate to spherical stars.

The process of rectification is open to criticism on several grounds. The first, and most obvious, is the presence of the sine terms in the Fourier series for $\ell(\theta)$. The presence of such terms should be justifiable from a physical point of view, but the present author knows of no

such justification, published or unpublished. Another criticism, raised by Kopal (1979, pg. 192), is that one is using a Fourier series outside its range of validity, since a series, which has been fitted to the out-of-eclipse observations is being applied to all observations, both in- and out-of-eclipse. Rectification will work for systems in which distortion effects and the reflection effects are minimal. The example in Appendix 1 is of this variety. In cases such as these, the B_n-terms are quite small in comparison to the A_n-terms. However, one really cannot apply rectification to highly distorted systems (very close binaries, e.g. W Ursae Majoris). In systems such as these, the shapes of the stars depart greatly from an ellipsoidal form, and actually approach a Roche-surface form. The theory upon which rectification rests is clearly not designed with such systems in mind. Consequently, rectification is no longer used, and more physically acceptable procedures have replaced it.

2.6 Differential Corrections

If the geometric elements r_1, r_2, i, L_1, L_2, x_1, and x_2 are well-determined (in the sense that the solution for these elements is determinate), one may improve the values of these elements by the 'differential corrections' procedure. Differential corrections are based on the idea that if

$$\ell(\theta) = u - \alpha L \qquad (u = \ell(90^\circ)) \qquad (2.27)$$

then

$$\Delta \ell(\theta) = \Delta u - \alpha \Delta L - L \Delta \alpha$$

$$= \Delta u - \alpha \Delta L - L \sum_{j=1}^{N} \frac{\partial \alpha}{\partial x_j} \Delta x_j \quad , \qquad (2.28)$$

where x_j is one of the elements $r_1, r_2, i, x_1,$ or x_2. Equation (2.28) may now be regarded as an equation of condition, so that if this equation is written out for each (θ, ℓ) pair, one may solve the system of equations for Δx_j's, Δu, and ΔL by the least-squares method. The value of $\Delta \ell$ is found by subtracting the calculated value of ℓ from the observed value (i.e., an '0-c'). The various partial derivatives appearing in equation (2.28) have different forms according to the eclipse type. The form of α must also be chosen according to the eclipse type. The paper by Irwin (1947) describes the procedure of differential corrections in great detail, and tables of the various derivatives are provided in an appendix to the paper. The present author has written a number of computer programs for performing the differential corrections procedure using the values of the derivatives from Irwin's tables or values generated by the equations for the derivatives. Some of these programs may be found in Appendix 1. It should be noted that one cannot apply differential corrections to every eclipsing binary star, since, as mentioned earlier, a well-determined set of elements is required, as well as a large number of observations to make the least-squares method truly applicable. Least-squares differential corrections should not

be regarded as a 'black box' that always generates improved values of the elements, therefore it should be applied with some discretion. As Irwin mentions in the paper quoted earlier, least-squares is no substitute for good sense!

2.7 Conclusion

The discussion of the Russell model and the method of light curve analysis associated with it is now complete. This model of an eclipsing binary star is best applied to systems having spherical components, since any application to systems having distorted stars will inevitably lead to the use of rectification, the validity of which is in some doubt. The Russell-Merrill method is still used to provide a preliminary set of elements to be used in more advanced methods of light curve analysis. In short, the Russell-Merrill method is not the most fruitful one, since it is possible to derive much more information from a light curve. It would also be of some advantage to have a method of analysis tailored for use on an electronic computer.

CHAPTER 3

THE METHOD OF KITAMURA

3.1 Introduction

With the development of the electronic computer in the late 1950s, many workers in the field of close binary stars began looking for methods of analyzing eclipsing binary light curves that might be suitable for electronic computation. The programs that were developed all used the Russell model and the Russell method for computing the orbital elements. In general, the method of analysis used was one of Kopal's iterative methods. A good example of such a computer program can be found in the work of Jurkevich (1970). This preliminary analysis was usually followed by the application of a differential corrections program to obtain improved values of the elements. However, the problems of determinacy were still present (especially for partial eclipses), as well as the ever-present problem of allowing for the various 'proximity effects' present in close eclipsing binary stars. It was suggested by Kopal in 1959, that the problem of the determination of the geometric elements of an eclipsing binary might be more easily solved if one were to somehow make use of the Fourier transform of the light curve. Such a method was formulated by Kitamura (1965). It still used as a basis the Russell model of an eclipsing binary, with rectification being required for close eclipsing binaries. Kitamura's method

solved in part the problem of 'proximity effects', since the Fourier transform operation (like any form of integration) is a smoothing operation. The required Fourier transforms being easily calculated by an electronic computer. The process of determining the elements was then one of matching the Fourier transforms of the observed light curve with those computed from the Russell model. This is the essence of Kitamura's method. We now examine the method in detail.

3.2 Incomplete Fourier Transforms

As previously mentioned, Kitamura's method is based on the Russell model. The observed light from an eclipsing binary system at any phase θ is, according to the Russell model,

$$\ell(\theta) = 1 - \alpha L ,$$

where ℓ, α, and L are the quantities defined in chapter one. Following Kitamura (1965, pg. 30), the Fourier transform of $\ell(\theta)$ is

$$\int_0^{\theta_1} \ell(\theta)\, e^{in\theta}\, d\theta = \int_0^{\theta_1} e^{in\theta}\, d\theta - L \int_0^{\theta_1} \alpha\, e^{in\theta}\, d\theta \qquad (i = \sqrt{-1})$$

Separating the real and imaginary parts, we arrive at the 'incomplete' transforms of $\ell(\theta)$

$$S_n \equiv \int_0^{\theta_1} \ell(\theta) \sin n\theta\, d\theta = \int_0^{\theta_1} \sin n\theta\, d\theta - L \int_0^{\theta_1} \alpha \sin n\theta\, d\theta \qquad (3.1)$$

and

$$C_n \equiv \int_0^{\theta_1} \ell(\theta) \cos n\theta \, d\theta = \int_0^{\theta_1} \cos n\theta \, d\theta - L \int_0^{\theta_1} \alpha \cos \theta n \, d\theta$$

$$n = 0, 1, 2, \ldots$$

For the purpose of analyzing the light curve, the left hand sides of equations (3.1) are used while the right hand sides are used for the model computations. The specification of the limits of integration remains to be decided. Since $\ell(\theta)$ is symmetric about either minimum, the lower limit of integration is obviously $\theta = 0$ (assume that $\theta = 0$ specifies the minimum of $\ell(\theta)$). To determine the upper limit, one may choose any phase $\phi > \theta_1$, θ_1 being the phase angle of last contact (see figure 11). The effect of choosing ϕ arbitrarily will be accounted for later. It should be noted that in general, there will not be a data point at exactly $\theta = 0$, so any data point having a phase ε close to $\theta = 0$ will suffice. Equations (3.1) become:

$$S_n = \int_\varepsilon^\phi \ell(\theta) \sin n\theta \, d\theta \quad \text{and} \quad C_n = \int_\varepsilon^\phi \ell(\theta) \cos n\theta \, d\theta \,. \qquad (3.2)$$

Without going into excessive detail, Kitamura found it convenient to define three parameters simply related to S_n and C_n to connect the Fourier transforms to the geometric elements. These parameters are

$$F_1 = \frac{L_1 s_1}{L_1 c_1} \;, \quad F_2 = \frac{L_1 s_2}{L_1 c_2} \quad \text{and} \quad E = \frac{L_1 c_o}{1 - \lambda} \qquad (3.3)$$

where

$$L_1 c_o = \phi - \varepsilon\lambda - C_o \;, \qquad (L_1 s_o = 0\,)$$

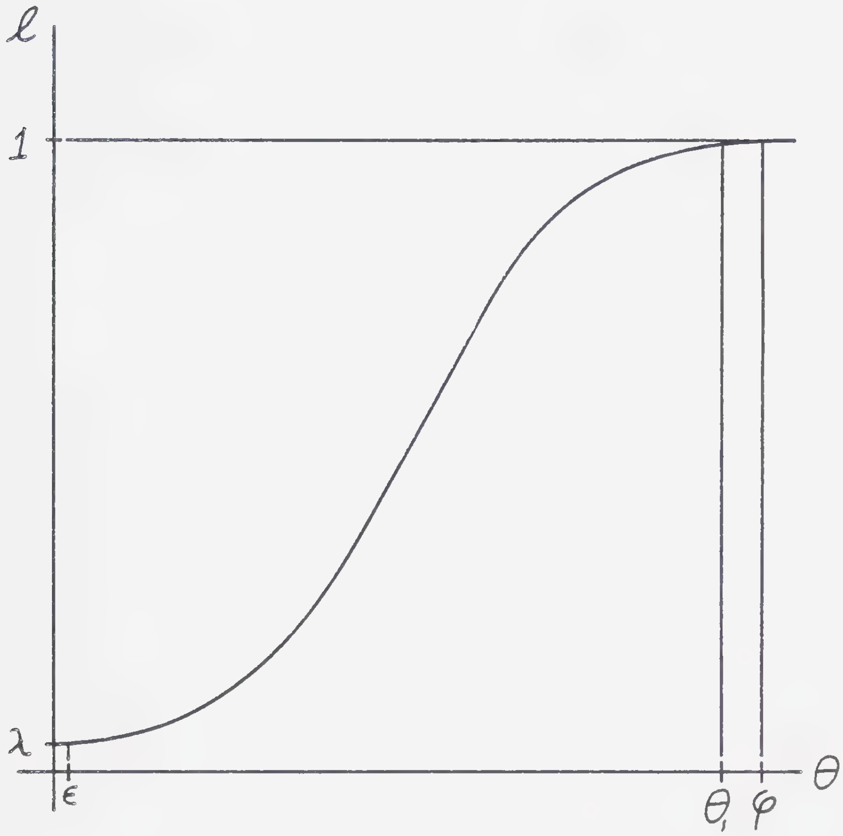


FIGURE 11. THE ANGLES $\varepsilon, \theta_1, \phi$.

$$L_1 s_n = \frac{1 - \cos n\phi}{n} - \frac{1 - \cos n\varepsilon}{n} \lambda - KS_n \quad , \qquad (3.4)$$

$$L_1 c_n = \frac{\sin n\phi}{n} - \frac{\sin n\varepsilon}{n} \lambda - KC_n \quad .$$

Once again, λ is the value of $\ell(\theta)$ at mid-eclipse, and K is a normalization factor for the light curve, which can be computed by

$$K = \frac{\phi_1 - \phi_2}{C_o(\phi_1) - C_o(\phi_2)} \quad ,$$

ϕ_1 and ϕ_2 being two phase angles outside the eclipse. The first two terms in each of equations (3.4) account for the choice of integration limits, ϕ and ε. The parameters F_1, F_2, and E may also be calculated with the right hand sides of equations (3.1). Kitamura (1967) performed such a computation, and the result was a large set of tables of F_1, F_2, and E for $0 < r_{1,2} \leq 1$ and $40^o \leq i \leq 90^o$. To account for limb darkening, Kitamura computed F_1, F_2, and E for zero and total limb darkening. He then constructed a table of 're-lated delta-functions', which allow interpolation for inter-mediate values of limb darkening. The 'related delta-functions' are defined by the following formulae:

$$\Delta'_{s_n} = \frac{s'_n - s^o_n}{s^o_n} \quad , \quad \Delta'_{c_n} = \frac{c'_n - c^o_n}{c^o_n} \quad \text{(occultation)} \; ,$$

$$\Delta''_{s_n} = \frac{3\Phi(k)s''_n - s^o_n}{2} \; , \; \Delta''_{c_n} = \frac{3\Phi(k)c''_n - c^o_n}{2} \quad \text{(transit)}, \quad (3.5)$$

$$\Phi(k) = \frac{4}{3\pi k^2} \sin^{-1}\sqrt{k} - \frac{\sqrt{k(1-k)}\,(12 + 8k - 32k^2)}{4\pi k^2}$$

where a prime denotes an occultation eclipse and a double prime a transit eclipse. For any darkening x, the characteristic functions become

$$F_n'^{x} = F_n^{o'} \frac{1 + x\Delta_{s_n}'}{1 + x\Delta_{c_n}'} , \quad F_n''^{x} = F_n^{o''} \frac{1 + x\Delta_{s_n}''}{1 + x\Delta_{c_n}''} , \quad E^x = E^o \frac{1 + x \quad c_o}{1 + \quad f(0)} , \tag{3.6}$$

$$f(0) = \frac{f(0)' - f(0)^o}{f(0)^o} , \quad f(\theta) = (1 - x)\alpha_o^o + \frac{3}{2}\, x\alpha_1^o$$
$$= f(0)^o[1 + x\Delta f(0)] ,$$

where

$$f(0)^o = \alpha_o^o \quad , \quad f(0)' = \frac{3}{2}\, \alpha_1^o \quad ,$$

f(0) being the light loss at mid-eclipse. If the left hand sides of these equations are taken to be the values found via equations (3.3), then if preliminary values of r_1, r_2, and i are known from an initial solution, the values of the Δ-functions may be found from the tables and (assuming a limb darkening x) used in equations (3.6) to solve for $F_n^{o'}$, $F_n^{o''}$ and E_o. Using these new values of the characteristic functions, one may find new values of r_1, r_2, and i.

3.3 Practical Approach

In practice, the incomplete Fourier transforms S_n and C_n are computed from the light curve by using the trapezoidal rule:

$$S_n = \frac{1}{2} \sum_{i=0}^{k-1} [\ell(\theta_i) \sin n\theta_i + \ell(\theta_{i+1}) \sin n(\theta_{i+1})][\theta_{i+1} - \theta_i]$$

and

$$C_n = \frac{1}{2}\sum_{i=0}^{k-1} [\ell(\theta_i)\cos n\theta_i + \ell(\theta_{i+1})\cos n(\theta_{i+1})][\theta_{i+1} - \theta_i]$$

To obtain improved values of S_n and C_n, end corrections to the trapezoidal rule may be used:

$$\Delta C_o = 0 \ , \quad \Delta C_1 = \frac{1}{12} h^2 \ell(\phi) \sin\phi \ , \quad \Delta C_2 = \frac{1}{6} h^2 \ell(\phi) \sin 2\phi \ ,$$

$$\Delta S_1 = \frac{1}{12} h^2 [\ell(0) - \ell(\phi)\cos\phi] \quad , \quad \Delta S_2 = \frac{1}{6} h^2 [\ell(0) - \ell(\phi)\cos 2\phi]$$

With these values of S_n and C_n, the values of $L_1 s_n$ and $L_1 c_n$ in equations (3.4) may be found, and hence the values of F_1, F_2, and E. The preceding analysis is performed for each observed eclipse, provided that the eclipses are deep enough to allow S_n and C_n to be computed. If the eclipse type is in doubt, the following 'rule of thumb' may be used:

$$[F_n]^{tr} \leq [F_n]^{oc} \quad .$$

This rule need only be applied in the case of partial eclipses, when the type of eclipse cannot be determined directly from the light curve. One can now consult Kitamura's tables of F_1, F_2, E and F_1/F_2 to find the appropriate elements r_1, r_2, and i. In general, one will end up with several sets of elements for which the values of the characteristic functions match. To determine which set of elements is the 'best fit', one can examine the values of T-1, where T is given by:

$$T = L_p + L_s\,,\quad L_p = \frac{D_p}{(c_o)_p}\,,\quad L_s = \frac{D_s}{(c_o)_s}\,,\quad (p = \text{primary eclipse},$$

$$D_p = (L_1 c_o)_p\,,\ D_s = (L_1 c_o)_s\,,\quad s = \text{secondary eclipse})$$

where the values of D_p and D_s can be computed from the first of equations (3.4), and $(c_o)_p$ and $(c_o)_s$ can be found from Kitamura's tables ($c_o^{oc} = E^{oc} f_a$, $c_o^{tr} = c_o^{oc} Q$ - both f_a and Q are listed in the tables adjacent to each set of elements r_1, r_2, and i). The set of elements for which T-1 is a minimum ($\simeq 0$) is the 'best' set of elements. If limb darkening is to be taken into account, the procedure described earlier, involving the related Δ-functions, may be used. The present author has written a computer program 'LCFT2', which computes S_n, C_n, F_1, F_2, F_1/F_2 and E for a given eclipse light curve. This program, along with a sample run, may be found in Appendix 2.

As with the Russell method, a differential corrector program may be used to improve the values of the best set of elements. In fact, a differential corrector should be used since the values of the elements obtained are read directly from the table and therefore represent only an approximate set of elements. The programs for differential corrections discussed in chapter one may be used for this purpose.

3.4 Conclusions

Kitamura's method has many advantages over Russell's method, since it uses all the data available instead of only

three or more fixed points. Furthermore, as mentioned earlier, the Fourier transform operation acts as a smoothing operation to reduce the effect of random errors in the observations. This aspect of Kitamura's method also has the advantage of speed and computational simplicity. Once the incomplete Fourier transforms have been found, it is just a matter of consulting the appropriate tables to get the values of the elements. The bulk of the computation is done by the computer.

However, there are distinct disadvantages in Kitamura's method. The first is that it is based on the Russell model of an eclipsing binary, so rectification is required for those cases in which the stars involved are significantly distorted, and where other effects (reflection, mass transfer) are present. Kitamura's method would be much more useful if it were based on a better (more realistic) model of an eclipsing binary. A second disadvantage is the need to consult a rather large set of tables in the last step in the analysis of a light curve. This procedure is not objectionable when results from both minima are available, or when a single minimum provides determinate results. A search through the tables in cases such as this takes only five or ten minutes at the most. However, in an indeterminate case, the researcher is faced with the prospect of wandering through 241 large-format (about 30 cm $\times$ 50 cm) pages to look for one set of elements. One can easily spend an afternoon doing this. All objections

aside, Kitamura's method provides an excellent way in which to compute the geometric elements of a well-detached (no photometric complications) eclipsing binary. From a historical point of view, Kitamura's method provides a starting point for an alternative form of light curve analysis, which culminated in the recent work of Kopal (1979).

CHAPTER 4

ANALYSIS OF LIGHT CURVES IN THE FREQUENCY DOMAIN

4.1 Introduction

To make further progress in the analysis of the light curves of eclipsing binary stars, one must dispose of the Russell model entirely, and adopt a more realistic, and consequently a more physically complex model of a binary star. Briefly, such a model would take into account details such as non-sphericity (e.g., using the Roche model), stellar atmospheres (for the purposes of deducing limb darkening, gravity darkening), and any other effects that might be present. Consequently, the mathematical machinery required grows in complexity.

A generalized approach, such as the one just described, has been developed by Kopal. A convenient summary of this work can be found in his recent book (1979). Briefly, Kopal considers eclipsing binaries with both spherical and non-spherical component stars from both a dynamical and a geometric (i.e., eclipse geometry) point of view. One can, therefore, look at an eclipsing binary in any degree of complexity, from a simple preliminary analysis to one taking into account the unique features of a particular binary star. Kopal achieves this flexibility by using the Fourier transform of the light curve, but in a way that is much different from the approach used by Kitamura. The details of Kopal's use of the Fourier transform of the

light curve will be expounded later. The motivation behind this approach is twofold: the Fourier analysis, in certain cases, renders the solution for the geometric elements (r_1, r_2 and i) algebraic, in the sense that the end result is an equation stating "$r_1 = \ldots$", for example. Furthermore, 'proximity' effects present in close binaries can be separated from light variations due to eclipses by the nature of their frequency spectra. Eclipse variations produce a continuous frequency spectrum, while proximity effects produce only discrete frequencies. In short, Kopal's approach provides both a realistic description of an eclipsing binary, and a relatively simple method for determining the geometric elements, given the Fourier transform of the light curve.

4.2 The Equations of Kopal's Method

To begin our exploration of Kopal's method, let us consider the Fourier transform of a light curve. The procedure and notation follow that of Kopal (1979). Consider first the light curve drawn in $\ell - \sin^{2m}\theta$ $(m = 1,2,3,\ldots)$ coordinates. The motivation for doing this will become clear later. Figure 12 shows the light curve plotted in the $\ell - \sin^{2m}\theta$ coordinates. The case of spherical stars will be considered first. There is a straightforward generalization for non-spherical stars. Let the area bounded by the light curve and the lines $\sin^{2m}\theta = 0$, $\ell = 1$ be denoted by A_{2m}. This area, known as the m-th moment of the light curve, is evidently given by

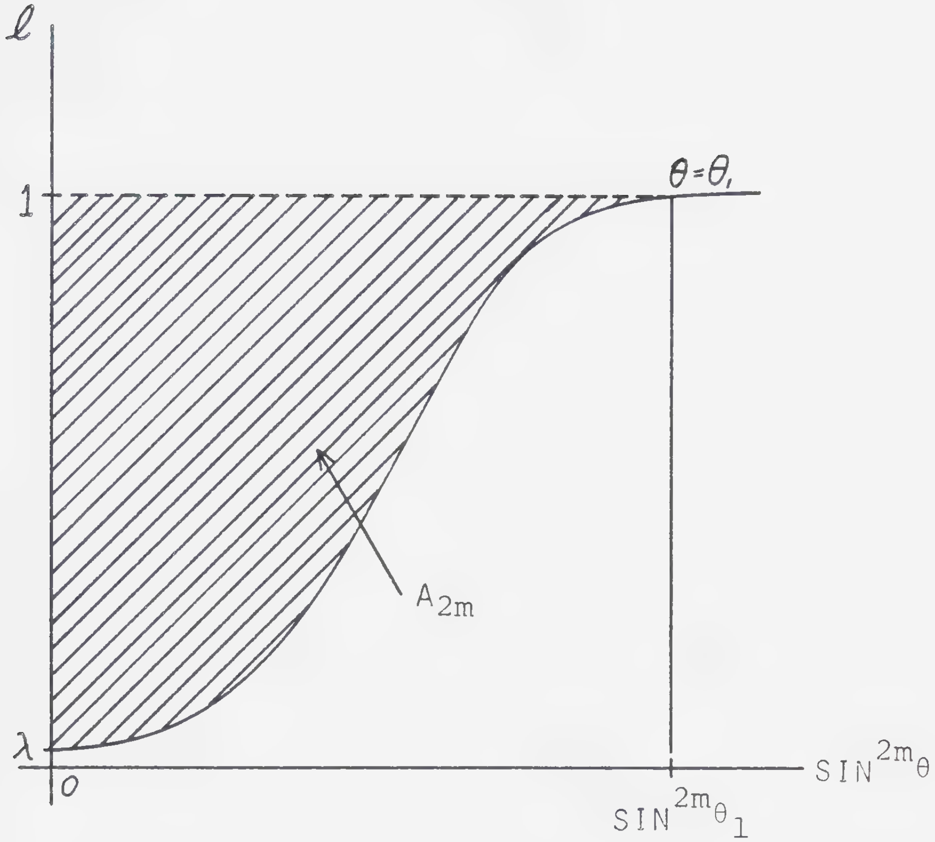


FIGURE 12. THE DEFINITION OF A_{2m}.

$$A_{2m} = \int_0^{\theta_1} (1-\ell)d(\sin^{2m}\theta) , \quad (m = 1,2,3,\ldots) \tag{4.1}$$

where θ_1 is the phase angle of last contact. The upper limit θ_1 was chosen because $1-\ell$ equals zero for $\theta > \theta_1$, giving zero contribution to the integral. By the very nature of the integral defining A_{2m}, one might suspect that A_{2m} is related to the Fourier transform of $1-\ell(\theta)$. Indeed, it is not difficult to show this. To proceed, the following result is required (Oberhettinger,1973; p. 31):

$$\sin^{2m}\theta = \frac{(2m)!}{4^m} \sum_{j=0}^{m} \frac{(-1)^j \varepsilon_j \cos 2j\theta}{(m+j)!(m-j)!} \tag{4.2}$$

where

$$\varepsilon_j = \begin{cases} 1 & j = 0 \\ 2 & j > 0 \end{cases} .$$

The differential of this series is, after a little algebra:

$$d(\sin^{2m}\theta) = \frac{(2m)!}{4^{m-1}} \sum_{j=1}^{\infty} \frac{(-1)^{j+1} j \sin 2j\theta}{(m+j)!(m-j)!} . \tag{4.3}$$

Multiplying by $1-\ell$, integrating from 0 to θ_1, and interchanging the order of summation and integration on the right-hand side, we arrive at

$$A_{2m} = \int_0^{\theta_1} (1-\ell)d(\sin^{2m}\theta)$$

$$= \frac{(2m)!}{4^{m-1}} \sum_{j=1}^{\infty} \frac{(-1)^{j+1} j}{(m+j)!(m-j)!} \int_0^{\theta_1} (1-\ell) \sin 2j\theta d\theta . \tag{4.4}$$

To relate this result to the Fourier transform of $(1-\ell)$, consider an arbitrary function $f(\theta)$ (which could be $1-\ell(\theta)$),

which is an even function (i.e., $f(-\theta) = -f(\theta)$). The Fourier transform of $f(\theta)$ will be

$$F(\nu) = \int_{-c}^{c} f(\theta)\, e^{-2\pi i\nu\theta}\, d\theta \ , \tag{4.5}$$

where ν is the spectral frequency, and $\pm c$ are the limits of integration, which can be infinite. In the situation at hand, the limits of integration will be the phases of first and last contact since $f(\theta) \equiv 1-\ell(\theta)$, which will be zero outside the eclipses ($|\theta| > c$). Let us now split up $f(\theta)$ in the following way:

$$f(\theta) = \frac{1}{2}\,[f(\theta) + f(-\theta)] + \frac{1}{2}[f(\theta) - f(-\theta)] \ .$$

The first term of this equation is an even function, while the second term is odd. Knowing this, it is not difficult to show that

$$\int_{-c}^{c} \frac{1}{2}[f(\theta) + f(-\theta)]\, e^{-i2\pi\nu\theta} d\theta = \int_{0}^{c} [f(\theta) + f(-\theta)] \cos\, 2\pi\nu\theta\, d\theta$$

$$\equiv F_1(\nu)$$

and (4.6)

$$\int_{-c}^{c} \frac{1}{2}\,[f(\theta) - f(-\theta)]\, e^{-i2\pi\nu\theta} d\theta = \int_{0}^{c} [f(\theta) - f(-\theta)] \sin 2\pi\nu\theta\, d\theta$$

$$\equiv F_2(\nu) \ .$$

The Fourier transform $F(\nu)$ can then be decomposed into its real and imaginary parts as $F_1(\nu) - iF_2(\nu)$. Now assume that $f(-\theta) = -f(\theta)$, so that $F_1(\nu)$ is zero and (letting $c = \theta_1 =$

phase of last contact)

$$F_2(\nu) = 2\int_0^{\theta_1} f(\theta)\sin 2\pi\nu\theta\, d\theta \quad . \tag{4.7}$$

For $f(\theta) = 1-\ell(\theta)$ and $\nu = j/\pi$ one has for $F_2(\nu)$:

$$\frac{1}{2} F_2\left(\frac{j}{\pi}\right) = \int_0^{\theta_1} (1-\ell)\sin 2j\theta\, d\theta \quad . \tag{4.8}$$

This integral is identical to the integral in the series expansion for A_{2m} derived earlier, and is also the expression for the Fourier coefficient b_{2j} in the Fourier sine series

$$1 - \ell = \sum_{j=1}^{\infty} b_{2j}\sin 2j\theta \quad . \tag{4.9}$$

This series is a special case of the more generalized Fourier series

$$f(\theta) = \frac{F_1(0)}{2c} + \frac{1}{c}\sum_{n=1}^{\infty} F_1\left(\frac{n}{2c}\right)\cos\frac{n\pi\theta}{c} + F_2\left(\frac{n}{2c}\right)\sin\frac{n\pi\theta}{c} \ , \quad \left(c = \frac{\pi}{2}\right) \tag{4.10}$$

when the upper limit of integration in $F_2(\nu)$ is extended to $\pi/2$. The Fourier coefficients b_{2j} are then given by

$$b_{2j} = \frac{2}{\pi} F_2\left(\frac{j}{\pi}\right) = \frac{4}{\pi}\int_0^{\pi/2} (1-\ell)\sin 2j\theta\, d\theta \quad . \tag{4.11}$$

Finally,

$$A_{2m} = \frac{(2m)!}{4^m}\sum_{j=1}^{m} \frac{\pi(-1)^{j+1} j}{(m+j)!(m-j)!}\, b_{2j} \quad . \tag{4.12}$$

This result shows the relationship that exists between the

A_{2m} and the Fourier transform of the light curve. The moments of the light curve, the A_{2m}'s, are essentially 'weighted means' of the Fourier coefficients b_{2j}. For the cases of interest in light curve analysis, $m = 1,2,3$. The A_{2m}'s are then

$$\begin{aligned} A_2 &= \frac{\pi}{4} b_2 \\ A_4 &= \frac{\pi}{4} (b_2 - \frac{1}{2} b_4) = A_2 - \frac{\pi}{8} b_4 \\ A_6 &= \frac{\pi}{4} (\frac{15}{16} b_2 - \frac{3}{4} b_4 + \frac{3}{16} b_6) \ . \end{aligned} \tag{4.13}$$

A point of particular importance here is that the moments A_{2m} are related to the coefficients b_{2j}, and these correspond to discrete frequencies in the Fourier sine series (4.9). The index m on the A_{2m}'s will then correspond to the frequencies used in the spectral analysis of the light curve.

Another feature worthy of note is that equation (4.1) is valid only for spherical stars. Furthermore, it applies for any value of limb darkening and any eclipse type. The computation of the A_{2m}-functions will be considered later.

The results just obtained may be generalized to the case in which the stars constituting the eclipsing binary are non-spherical. From the point of view of data analysis, this step is quite simple. One need only replace the integrand of equation (4.1) (that is, the function $1-\ell(\theta)$) by $\ell(\pi/2) - \ell(\theta)$, where $\ell(\pi/2)$ is the value of $\ell(\theta)$ at quadrature. In the case of spherical stars, $\ell(\pi/2) = \ell(\theta_1) = 1$,

the upper limit of integration being θ_1. In the case at hand, however, the upper limit of integration is extended to $\theta = \pi/2$ since $\ell(\pi/2) - \ell(\theta)$ is, in general, non-zero for $0 \leq \theta \leq \pi/2$. Furthermore, since it is often difficult to separate the eclipse phases from the non-eclipse phases in non-spherical eclipsing binaries, an extension of the upper limit of integration to $\theta = \pi/2$ will adequately cover any eclipse phases (between $\theta = 0$ and $\theta = \pi/2$) that might be present. The integral defining A_{2m} becomes:

$$A_{2m} = \int_0^{\pi/2} [\ell(\tfrac{\pi}{2}) - \ell(\theta)]d(\sin^{2m}\theta) \ . \tag{4.14}$$

For future reference, this particular integral will be denoted by $\bar{A}_{2m}$ to distinguish it from its spherical geometry counterpart, equation (4.1).

To implement Kopal's method, one must be able to determine A_{2m} or $\bar{A}_{2m}$ (whichever is appropriate) from the observed light curve of an eclipsing binary. There are several methods available for computing A_{2m} or $\bar{A}_{2m}$, given $\ell(\theta)$.

The most straightforward approach is to numerically evaluate equations (4.1) and (4.14) by the trapezoidal rule. If one samples the light curve at equally-spaced points, the standard trapezoidal rule applies:

$$\int_0^{\theta_1} (1-\ell)d(\sin^{2m}\theta) = \int_0^{\theta_1} (1-\ell)\ 2m\sin^{2m-1}\theta\ d\theta \tag{4.15}$$
$$= 2m[\tfrac{h}{2}((1-\ell_1)\sin^{2m}\theta_1 + (1-\ell_N)\sin^{2m}\theta_N) + \sum_{i=2}^{N-1}(1-\ell_i)\sin^{2m}\theta_i],$$

where N is the number of observations, $(\theta_i , 1-\ell_i)$ are the N data points, and $h = \theta_1/N$ is the sampling interval for the integral. Alternatively, the data may be used directly in the following integration rule (Niarchos, 1981)

$$A_{2m} = \sum_{i=1}^{N} [\ell(\tfrac{\pi}{2}) - \tfrac{1}{2}(\ell_i+\ell_{i+1})][\sin^{2m}\theta_{i+1} - \sin^{2m}\theta_i] \ . \qquad (4.16)$$

Similar formulae hold for the evaluation of $\bar{A}_{2m}$. The process of numerical integration just described is very useful and is adequate for most purposes. However, there are some drawbacks. First, one is unable to perform any sort of error analysis if the moments of the light curve are formed by numerical integration. The various aspects of error analysis will be explored later. Moreover, in the case of the $\bar{A}_{2m}$'s, a lack of points near $\theta = \pi/2$ will seriously affect the value of the integral. This effect has been noticed by Koul and Abhyankar (1982) and also by the present author in earlier work (1983). This problem arises from the fact that the value of $\ell(\theta)$ closest to $\theta = \pi/2$ gives small but significant contributions to $\bar{A}_{2m}$. Since $\ell(\theta)$ is increasing in the range $0 \leq \theta \leq \pi/2$, one would expect a large contribution to $\bar{A}_{2m}$ from values of $\ell(\theta)$ near $\theta = 0$, but steadily smaller contributions as θ approaches $\pi/2$. This problem is illustrated in figure 13. This effect is particularly prevalent when an observer has concentrated on acquiring a large number of eclipse observations, but has paid relatively little attention to the light curve outside the eclipses.

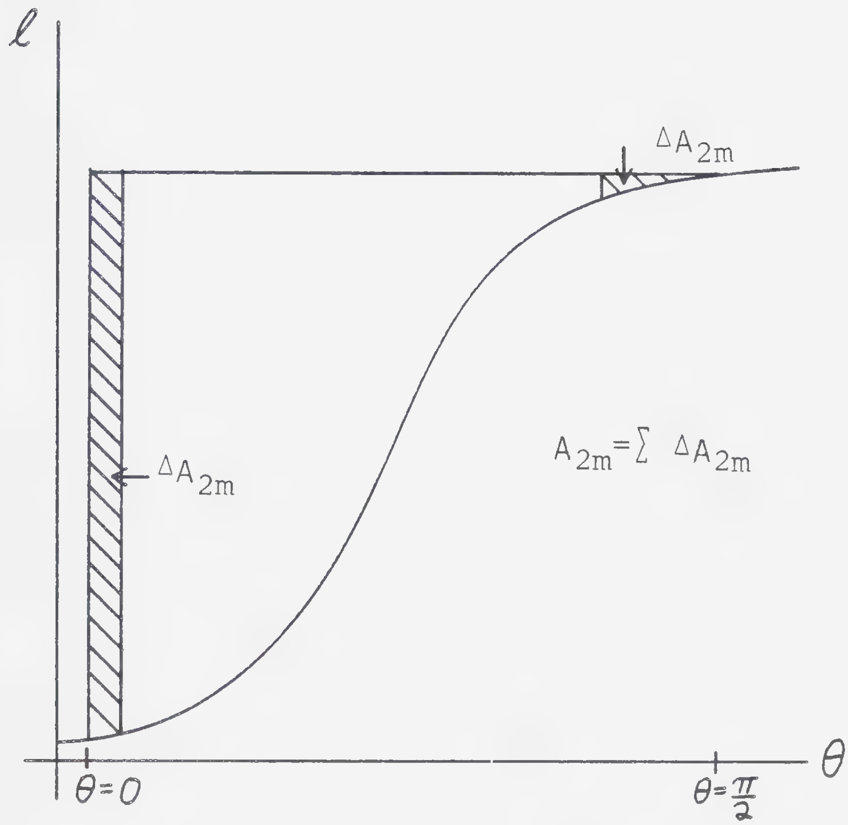


FIGURE 13. CONTRIBUTIONS TO A_{2m}.

4.3 The Use of Fourier Series

A much more suitable method for determining the moments of the light curve is to fit a Fourier cosine series to $1-\ell(\theta)$ or $\ell(\pi/2)-\ell(\theta)$. In the spherical star case, one can fit $1-\ell$ directly with a Fourier series of the form

$$1 - \ell = \frac{1}{2} a_o + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi\theta}{\theta_1}\right) \tag{4.17}$$

where θ_1 = phase of last contact and N is the maximum number of terms in the series to be used. The cosine series is used because $1-\ell(\theta)$ is an even function of θ, hence all of the sine coefficients will be zero. To find the various Fourier coefficients a_n $(n = 0,1,\ldots,N)$ in equation (4.17), one must first estimate θ_1, and then form equation (4.17) for each observation in the range $0 < \theta < \theta_1$. The resulting system of equations may be solved by the least-squares method. This process for determining the Fourier coefficients is described in some detail by Kopal (1982a). Since the least-squares solution can also produce estimates of the uncertainties of the Fourier coefficients, an error analysis may be done. Before venturing into this, the relationship between the Fourier coefficients and the moments of the light curve A_{2m} should be explored. Following the development given by Kopal (1979, pg. 241), let us begin with the following identity (Oberhettinger, 1973; p. 31)

$$\sin^{2m}\theta = \frac{\Gamma(2m+1)}{2^{2m-1}} \sum_{j=0}^{\infty} \frac{(-1)^j \sin(2j+1)\theta}{\Gamma(\frac{2m+3}{2} + j)\Gamma(\frac{2m+1}{2} - j)} \; , \quad (0 \leq \theta \leq \pi) \; .$$

As with equation (4.4) of this chapter, differentiate both sides of this equation, multiply by $1-\ell$, and integrate between $\theta = 0$ and $\theta = \theta_1$:

$$A_{2m} \equiv \int_0^{\theta_1} (1-\ell)\,d(\sin^{2m}\theta)$$
$$= \frac{\Gamma(2m+1)}{2^{2m-1}} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(m+j+\frac{3}{2})\,\Gamma(m-j+\frac{1}{2})} \int_0^{\theta_1} (1-\ell)\,d[\sin(2j+1)\theta] \; . \tag{4.18}$$

Kopal has evaluated the integral on the right-hand side of equation (4.18) in terms of the Fourier coefficients a_n. The procedure used is not unlike that used to derive equation (4.12) earlier. Without going into the algebraic details, the following result is obtained (Kopal (1979, pg. 241):

$$A_{2m} = \frac{\Gamma(2m+1)}{4^m} \sum_{j=0}^{\infty} \frac{(-1)^j \sin(2j+1)\theta_1}{\Gamma(m+j+\frac{3}{2})\,\Gamma(m-j+\frac{1}{2})} \sum_{n=0}^{\infty} \frac{(-1)^n[(2j+1)\theta_1]^2}{[(2j+1)\theta_1]^2 - [n\pi]^2}\, a_n \; . \tag{4.19}$$

The number of terms to be used in the second sum in equation (4.19) will depend upon how many coefficients can be determined significantly from the data. Typically, one can determine only five or six coefficients from the least-squares solution of equation (4.17). To facilitate the determination of the A_{2m}'s from the Fourier coefficients a_n, Kopal (1982a, pp. 131-132) summed the first series in equation (4.19) explicitly, leading to much more tractable relations for the A_{2m}'s $(m = 0,1,2,3)$:

$$A_o = \frac{1}{2} a_o + a_1 + a_2 + \ldots + a_N$$

$$A_2 = \sum_{\nu=0}^{\infty} \left(\frac{\varepsilon_\nu}{2} \frac{a_2 \sin^2\theta_1}{1 - [\nu\pi/\theta_1]^2} + \frac{a_{2\nu+1} \cos^2\theta_1}{1 - [(2\nu+1)\pi/2\theta_1]^2} \right)$$

$$A_4 = \sum_{\nu=0}^{\infty} \frac{\varepsilon_\nu}{2} \left(\frac{\sin^2\theta_1}{1 - [\nu\pi/\theta_1]^2} - \frac{\sin^2 2\theta_1}{4 - [\nu\pi/\theta_1]^2} \right) a_{2\nu} +$$

$$+ \sum_{\nu=0}^{\infty} \left(\frac{\cos^2\theta_1}{1 - [(2\nu+1)\pi/2\theta_1]^2} - \frac{\cos^2 2\theta_1}{4 - [(2\nu+1)\pi/2\theta_1]^2} \right) a_{2\nu+1} \tag{4.20}$$

$$A_6 = \frac{3}{16} \sum_{\nu=0}^{\infty} \frac{\varepsilon_\nu}{2} \left(\frac{5 \sin^2\theta_1}{1 - [\nu\pi/\theta_1]^2} - \frac{8 \sin^2 2\theta_1}{4 - [\nu\pi/\theta_1]^2} + \right.$$

$$\left. + \frac{3 \sin^2 3\theta_1}{9 - [\nu\pi/\theta_1]^2} \right) a_{2\nu} + \frac{3}{16} \sum_{\nu=0}^{\infty} \left(\frac{5 \cos^2\theta_1}{1 - [(2\nu+1)\pi/2\theta_1]^2} - \right.$$

$$\left. - \frac{8 \cos^2 2\theta_1}{4 - [(2\nu+1)\pi/2\theta_1]^2} + \frac{3 \cos^2 3\theta_1}{9 - [(2\nu+1)\pi/2\theta_1]^2} \right) a_{2\nu+1} ,$$

where

$$\varepsilon_\nu = \begin{cases} 1 & \nu=0 \\ 2 & \nu>0 \end{cases} .$$

The present author has adapted a program published by Jurkevich (1981), which performs the Fourier analysis of the light curve and the computation of the moments. Examples of this analysis will appear in later chapters, when individual stars are considered. The process of least-squares Fourier analysis is quite sufficient for the analysis of most eclipsing binary stars, but it must be generalized if it is to be applied to very close eclipsing systems, in which the stars are severely distorted (e.g.

W Ursae Majoris-type systems). This generalized form of Fourier analysis has been developed by Kopal (1982b). The mathematical development of Kopal's generalized Fourier analysis is quite complicated, and only the results will be stated here. The method also employs the dynamical theory of close binary stars, which will be considered later in this chapter. One begins by writing $1-\ell$ as a Fourier cosine series valid in the range $-\pi/2 \leq \theta \leq \pi/2$:

$$1 - \ell = \frac{1}{2} \sum_{m=0}^{M} \varepsilon_m a_m \cos 2m\theta \quad , \tag{4.21}$$

where ε_m is the factor defined in equation (4.20), and M is the number of coefficients to be used. Once again, the least-squares method is used to determine the Fourier coefficients a_m. The next step in the analysis is to determine the 'modulated moments' $\tilde{B}_n^{(\lambda)}$ from

$$\tilde{B}_n^{(\lambda)} = \sum_{m=0}^{M} \phi_m^{(\lambda,n)} a_m \qquad (\lambda = 1,3,5, \quad n = 3,4). \tag{4.22}$$

The $\phi_m^{(\lambda,n)}$-functions have been tabulated by Kopal (1982b, pp. 447-448). The modulated moments $\tilde{B}_n^{(\lambda)}$ may now be related to the moments of the light curve through

$$\tilde{B}_n^{(\lambda)} = \sum_{j=1}^{\infty} \rho_j^{(\lambda,n)} A_{2j} \qquad (\lambda = 1,3,5, \quad n = 3,4), \tag{4.23}$$

where the $\rho_j^{(\lambda,n)}$-functions have been tabulated by Kopal (1982b, pg. 449). The A_{2j}'s may be found either by the method of least-squares or by setting up as many equations of the form (4.23) as there are A_{2j}'s to be determined (N

equations in N unknowns). Since only A_2, A_4, and A_6 are required, only three equations are required. However, A_o is still to be determined. A_o is now given by

$$A_o = 1 - \lambda \sum_{j=1}^{J} c_j \quad , \tag{4.24}$$

where

$$1 - \lambda = \frac{1}{2} a_o + a_1 + a_2 + \ldots + a_M \; .$$

The coefficients c_j are part of the light curve modulation process, and will be considered in more detail later. A further equation incorporating the c_j's is:

$$\tilde{B}_o^{(\lambda)} = \sum_{j=1}^{J} \frac{\lambda c_j}{\lambda + j} + \sum_{j=1}^{\infty} \rho_j^{(\lambda,o)} A_{2j} \quad . \tag{4.25}$$

Equation (4.25) may be used to solve for the c_j's (by matrix inversion), since $\tilde{B}_o^{(\lambda)}$ may be found from equation (4.22), and since the second summation in equation (4.25) contains $\rho_j^{(\lambda,0)} A_{2j}$, which can be determined for $j = 1,2,3$. A_o can now be evaluated via equation (2.24). This completes the determination of the moments of the light curve for the non-spherical case. The procedure just described is a 'filter' for removing proximity effects, namely ellipticity (non-sphericity) and reflection. If tidal distortion is important, further results from the dynamical theory of close binary stars may be used. As a final note, one may determine the error in the resulting values of the A_{2m}'s found via the process of generalized Fourier analysis. This is done with the aid of the following equation:

$$\varepsilon_n^{(\lambda)} = \sum_{j=n}^{\infty} K_j^{(\lambda,n)} c_j \tag{4.26}$$

where, once again, the $K_j^{(\lambda,n)}$ functions have been tabulated by Kopal (1982b, pp. 445-446). Equation (4.26) gives the error incurred by using $n = 3,4$ in equations (4.22) and (4.23). The values of the $\varepsilon_n^{(\lambda)}$'s will be quite small in most cases.

4.4 The Use of Numerical Integration

The moments of the light curves of non-spherical eclipsing binary stars may also be found by the technique of numerical integration (Kopal, 1979, p. 195) discussed earlier. Recall that (equation (4.14)),

$$\bar{A}_{2m} = \int_0^{\pi/2} [\ell(\tfrac{\pi}{2}) - \ell(\theta)]\, d(\sin^{2m}\theta)$$

was the expression for the moment of the light curve in the non-spherical case. As in the Fourier analysis technique, one may modulate the light curve and obtain the moments A_{2m} of the light curve for an equivalent 'spherical' system. To do this, the following equation must be used:

$$\bar{A}_{2m} = -m \sum_{j=1}^{4} B(m, \tfrac{1}{2}j+1)\, c_j + A_{2m} + B_{2m}\ , \tag{4.27}$$

where $B(m, \frac{1}{2}j+1)$ is the beta-function (equal to $\Gamma(m)\Gamma(\frac{1}{2}j+1)/\Gamma(m+\frac{1}{2}j+1)$), c_j are the dynamical coefficients mentioned earlier. The last term in (4.27), the B_{2m}, is known as a 'photometric perturbation', and describes the departure of the star's shape from the spherical form. These quantities

are usually defined by rather complex expressions. The general form of the B_{2m} term is (Kopal, 1979, p. 195):

$$B_{2m} \equiv L_1 \sum_{h=1}^{\Lambda+1} C^{(h)} \int_0^{\pi/2} [f_*^{(h)} + f_1^{(h)} + f_2^{(h)}] d(\sin^{2m}\theta) \tag{4.28}$$

where L_1 is the relative luminosity of the star being eclipsed, Λ is the degree of limb darkening, and the coefficients $C^{(h)}$ are defined by:

$$C^{(o)} = \frac{1 - u_1 - u_2 - \ldots - u_\Lambda}{1 - \sum_{\ell=1}^{\Lambda} \frac{\ell u_\ell}{\ell+2}}$$

and (4.29)

$$C^{(h)} = \frac{u_h}{1 - \sum_{\ell=1}^{\Lambda} \frac{\ell u_\ell}{\ell+1}}$$

the u_i's the coefficients of limb darkening (u_1 = linear darkening, u_2 = quadratic darkening, etc.). The term in square brackets in equation (4.28) describes the effects of rotational and tidal distortion. To determine the A_{2m}'s for $m = 1,2,3$, one computes the $\bar{A}_{2m}$'s by numerical integration, and the coefficients c_j by numerical integration via

$$c_j = \int_{-a}^{a} [\ell(\tfrac{\pi}{2}) - \ell(\theta)] p_j^{(a,n)}(x)\, dx \qquad (x = \cos\theta) \tag{4.30}$$

where $\pm a = \pm\cos\theta_1$ represent the limits of that part of the light curve,which is free from eclipses. $p_j^{(a,n)}(x)$ is a 'modulating polynomial', which describes a star's surface distortion to first order. Kopal presents these polynomials for $n = 4$, $j = 1$ to 4, and for various values of $a = \cos\theta_1$

(1979, pp. 198-203). Alternatively, one may evaluate the sum in equation (4.27) directly by using

$$m \sum_{j=1}^{n} B(m, \tfrac{1}{2} j+1) c_j = \int_{-a}^{a} [\ell(\tfrac{\pi}{2}) - \ell(\theta)] Q_m^{(a,n)}(x)\, dx \quad , \qquad (4.31)$$

where the $Q_m^{(a,n)}(x)$ are polynomials similar to the $p_j^{(a,n)}(x)$ polynomials. At this point, one would evaluate the B_{2m}-term using an expression appropriate for the eclipse type. Kopal gives B_{2m} for $m = 1,2,3$ for the case of a total eclipse (1979, p. 211). However, complete expressions for all types of eclipses may be found in the pages by Livanion (1977) and Rovithis-Livanion (1979). With the B_{2m}-term evaluated, the process of finding A_{2m} is one of simple algebra. Once again, it should be emphasized that this particular method of determining the A_{2m}'s is useful only if the light curve has been well-observed, particularly around $\theta = \pi/2$.

4.5 Computing the A_{2m}'s with the Kalman Filter

An approach which is entirely different from either the Fourier analysis or numerical integration techniques for finding the A_{2m} is the Kalman filter algorithm. This approach was first formulated by Jurkevich (1976). In essence, the Kalman filter provides a means of determining the values of the A_{2m} recursively, while producing a result that is optimal in the least-squares sense. It combines the features of numerical integration and least-squares fitting, giving a fast and compact algorithm for determining

the A_{2m}'s. The theory of Kalman filtering has been investigated in detail, and a particularly good discussion is provided in the book by Bryson and Ho (1969).

In order to use the Kalman filter, one must be able to write the equations of a problem as ordinary differential equations. These differential equations are then treated as 'stochastic differential equations', or equations in variables which are statistical in nature (random variables having a variance σ^2 and a mean equal to zero). These differential equations are then integrated between the observations, resulting in a set of difference equations for the variables involved. In the problem at hand, the variables are A_{2m} (hereafter referred to as μ_m), $x = 1-\ell$, and the errors in the various μ_m and x, here denoted by p_m and p_x. The treatment to be followed here is that given by Jurkevich (1976). The differential equations of the problem are:

$$\dot{\mu}_m(t) = 2mx(t)\sin^{2m-1} t \cos t \, dt \,, \quad \mu_m(0) = 0$$

$$\dot{x}(t) = 0 \,, \qquad x(0) = 1 - \ell(0)$$

$$\dot{p}_x(t) = q \,, \qquad p_x(0) = r$$

$$\dot{p}_m(t) = 2m \sin^{2m-1} t \cos t \, A_x$$

where t is the phase, and q serves to connect the stochastic function with the function being estimated. The transfer to stochastic variables is very simple: stochastic variables will be denoted by a variable with a hat, i.e.,

$\hat{\mu}_m$. If the differential equations above (in their stochastic form) are integrated between observations with phases $t_{i+1} = b$ and $t_i = a$, the following recursion relations are obtained:

$$\left.\begin{aligned}
\hat{x}_{i+1} &= \hat{x}_i \\
\hat{\mu}_{i+1} &= \hat{\mu}_{m(i)} + \hat{x}_i[\sin^{2m} b - \sin^{2m} a] \\
p_{x(i+1)} &= p_{x(i)} + q(b-a) \\
p_{m(i+1)} &= p_{m(i)} + p_{x(i)}[\sin^{2m} b - \sin^{2m} a] + \\
&\qquad q[(b-a)\sin^{2m} b - f_m(b) - f_m(a)]
\end{aligned}\right\} \qquad (4.32a)$$

$$\left.\begin{aligned}
p_{x+1} &= [p_{x(i+1)}^{-1} - r^{-1}]^{-1} \\
p_{m(i+1)} &= p_{m(i+1)}[1 + p_{x(i+1)}/r]^{-1} \\
\hat{x}_{i+1} &= \hat{x}_{i+1} + \frac{p_{x(i+1)}}{r}[z_i - \hat{x}_i] \\
\hat{\mu}_{m(i+1)} &= \hat{\mu}_{m(i+1)} + \frac{p_{m(i+1)}}{r}[z_i - \hat{x}_i]
\end{aligned}\right\} \qquad (4.32b)$$

Here, r is the mean square value of the observational errors. Equations (4.32a) serve to 'propagate' the solution from one observation to the next, while equations (4.32b) 'update' the solution by processing the latest data point. Jurkevich (1981) has published a FORTRAN program which uses the Kalman filter algorithm to compute A_{2m} for $m = 1,2,3,4$. The program itself is very simple, with all the computations being contained in one DO-loop.

The present author has adapted this program for use on the MTS FORTRAN compiler at the University of Alherta. The only modifications required were the replacement of certain input and output lines. The program is listed in Appendix 3.

The Kalman filter has its advantages and disadvantages. As was mentioned earlier, the algorithm is compact and simple, and it has the desirable quality that the values of the A_{2m}'s obtained are least-squares estimates. The relative performance of the Kalman filter will be considered in a later chapter when individual eclipsing binaries are discussed. At this point, it suffices to say that the Kalman filter produces satisfactory values for the A_{2m}'s. A major disadvantage is that the error analysis produces spuriously low numbers. Jurkevich (1976, 1981) has investigated this, but the cause of the problem remains a mystery. A possible solution, suggested by Jurkevich (1980), is that the system of differential equations (4.32), which constitute a model of an eclipsing binary, are too simple for the purpose of error analysis.

4.6 Error Analysis for the A_{2m}'s

After the A_{2m}'s have been found by one of the procedures just outlined, one can perform an error analysis to give the uncertainties in the values of the A_{2m}'s. This error analysis may be done quite easily when least-squares Fourier analysis is used (i.e., equations (4.17) and (4.21)), since the least-squares solution produces the

uncertainties of the Fourier coefficients directly. However, error analysis is virtually impossible when numerical integration is used, since the error formulae for numerical integration rules are usually quite cumbersome to use (see, for example, Gerald, p. 211).

Demircan (1981, p. 127) has found an approximate expression for the error in the A_{2m}'s found by numerical integration:

$$\Delta A_{2m} \simeq \Delta u \sin^2 \theta_1 \quad , \tag{4.33}$$

where Δu is the error in ℓ at $\theta = \pi/2$ (quadrature), and θ_1 is the phase angle of first (or last) contact. In most cases, one is interested in only an approximate value for ΔA_{2m}, so equation (4.23) is sufficient for most purposes. To determine ΔA_{2m} when least-squares Fourier analysis is used is quite straightforward, one simply applies the rules of error propagation to equations (4.20). This results in the following expressions:

$$\Delta A_o = \frac{1}{2} \Delta a_o + \Delta a_1 + \Delta a_2 + \ldots$$

and

$$\Delta A_{2m} = \sum_{n=0}^{N} \phi_n^{(m)} \delta a_n \tag{4.34}$$

where

$$\phi_n^{(m)} = \frac{(2m)!}{4m} \sum_{j=0}^{\infty} \frac{(-1)^{j+n} \varepsilon_n \sin(2j+1)\theta_1}{\Gamma(m+j+\frac{3}{2})\Gamma(m-j+\frac{1}{2})} \frac{[(2j+1)\theta_1]^2}{(2j+1)\theta_1{}^2-[n\pi]^2} .$$

It should be noted that the unsimplified general expression for A_{2m} in terms of the Fourier coefficients has been used

in equation (4.33). If one were to use equation (4.20) directly, a typical result would be:

$$\Delta A_2 = \sum_{\nu=0}^{\infty} \left(\frac{\varepsilon_\nu}{2} \frac{\sin^2\theta_1}{1 - [\nu\pi/\theta_1]^2} \Delta a_2 + \frac{\cos^2\theta_1}{1 - [(2\nu+1)\pi/2\theta_1]^2} \right) \Delta a_{2\nu+1} , \tag{4.35}$$

with ΔA_4, ΔA_6 being found in a similar fashion. Clearly, only the errors in the Fourier coefficients are considered in equations (4.34) and (4.35).

Considering these results, there is a definite advantage to using least-squares Fourier analysis to determine the A_{2m} rather than numerical integration. The problem now is to relate the moments of the light curve (A_{2m}) to the geometric elements r_1, r_2, i, L_1, and L_2, for the cases of both spherical and non-spherical stars.

4.7 Computing the Elements

The problem now is to relate the moments of the light curve, the A_{2m} ($m = 0,1,2,3$), to the elements r_1, r_2, L_1, and i. To explore the relationship between the A_{2m}'s and the elements r_1, r_2, L_1, and i, the relatively simple case of a total eclipse of a uniformly bright star will be considered in detail. This will serve to introduce the more complicated cases in which limb darkening and distortion are present. In all cases, the method of solution will be outlined.

To relate the moments of the light curve to the geometric elements r_1, r_2, L_1, and i in the case of the

total eclipse of a uniformly bright star, following the treatment given by Kopal (1979, pp. 148-153), one starts with the equation defining the A_{2m} function:

$$A_{2m} = \int_0^{\theta_1} (1-\ell)\, d(\sin^{2m}\theta) \tag{4.36}$$

where, once again, θ_1 is the phase angle of fourth contact. In the case being considered here, one may write $1-\ell$ in terms of L_1 and $\alpha(k,p)$, which were defined in the chapter dealing with the Russell model. The equation is:

$$1-\ell(\theta) = \alpha(k,p)\, L_1 \quad . \tag{4.37}$$

The geometric relation must also be used to replace $\sin^{2m}\theta$:

$$\delta^2 = \sin^2\theta\, \sin^2 i + \cos^2 i \quad . \tag{4.38}$$

If $\delta_o \equiv \cos i$ (δ at $\theta = 0$), then solving for $\sin^2\theta$ gives:

$$\sin^2\theta = \frac{\delta^2 - \delta_o^2}{1 - \delta_o^2} \quad ,$$

or to the power m:

$$\sin^{2m}\theta = m(\delta^2 - \delta_o^2)^m (1 - \delta_o^2)^{-m} \quad .$$

Differentiating with respect to δ^2 gives

$$d(\sin^{2m}\theta) = m(\delta^2 - \delta_o^2)^{m-1} (1 - \delta_o^2)^{-m} d\delta^2 \quad .$$

Therefore, A_{2m} becomes (with $(1 - \delta_o^2)^{-m} = \csc^{2m} i$) :

$$A_{2m} = m\, L_1 \csc^{2m} i \int_{\delta_o^2}^{\delta_1^2} (\delta^2 - \delta_o^2)^{m-1}\, \alpha\, d\delta^2 \quad . \tag{4.39}$$

This equation is our starting point for the analysis of a total eclipse of a uniformly bright star. The limits δ_1 and δ_2 are the values of δ at first contact (corresponding to θ_1) and at second contact ($\theta = \theta_2$). At first contact, the two stars will appear to be "touching", so $\delta_1 = r_1 + r_2$. At second contact, the larger star (with radius r_2) will have just "covered" the smaller one (radius r_1), so $\delta_2 = r_2 - r_1$. Therefore, $\delta_1 > \delta_2 > \delta_o$ for a total eclipse. A_{2m} may be split up in the following way:

$$A_{2m} = m\,L_1 \csc^{2m} i \left(\int_{\delta_o^2}^{\delta_2^2} (\delta^2 - \delta_o^2)^{m-1} \alpha\, d\delta^2 + \int_{\delta_2^2}^{\delta_1^2} (\delta^2 - \delta_o^2)^{m-1} \alpha\, d\delta^2 \right). \tag{4.40}$$

The first integral is the contribution from the total phase of the eclipse, in which the smaller star is completely covered up, with the second integral describing the contributions from the partial eclipse phases. The first integral is quite easy if one remembers that $\alpha = 1$ during the total phase:

$$\int_{\delta_o^2}^{\delta_2^2} (\delta^2 - \delta_o^2)^{m-1} d\delta^2 = m^{-1} (\delta^2 - \delta_o^2)^m \Bigg|_{\delta_o^2}^{\delta_2^2}$$

$$= m^{-1} (\delta_2^2 - \delta_o^2)^m \quad .$$

Matters are not so simple during the partial phases. Doing the second integral by parts gives:

$$\int_{\delta_2^2}^{\delta_1^2} (\delta^2-\delta_o^2)^{m-1} \alpha \, d\delta^2 = m^{-1} (\delta^2-\delta_o^2)^{m-1} \alpha \Big|_{\delta_2}^{\delta_1} - m^{-1} \int_{\delta_2}^{\delta_1} (\delta^2-\delta_o^2)^m \frac{\partial\alpha}{\partial\delta} \, d\delta \ . \tag{4.41}$$

If an expression for $\partial\alpha/\partial\delta$ can be found, the integration will be complete. For a uniformly bright disk, α has the following form:

$$\pi r_1^2 \alpha = r_1^2 [\phi_1 - \tfrac{1}{2}\sin 2\phi_1] + r_2^2 [\phi_2 - \tfrac{1}{2}\sin 2\phi_2] \tag{4.42}$$

$$\cos\phi_1 = \frac{\delta^2 + r_1^2 - r_2^2}{2\delta r_1} \quad \text{and} \quad \cos\phi_2 = \frac{\delta^2 - r_1^2 + r_2^2}{2\delta r_2} \ . \tag{4.43}$$

Using these equations, the following expression for $\partial\alpha/\partial\delta$ results:

$$\frac{\partial\alpha}{\partial\delta} = -\frac{1}{\pi r_2} \left(\frac{r_2}{r_1}\right)^2 \sqrt{\frac{(\delta_1^2 - \delta^2)(\delta^2 - \delta_2^2)}{\delta^2 r_2^2}} \ .$$

This equation may be written in a simpler form if one introduces a new variable ϕ defined by

$$\delta^2 = r_1^2 - 2r_1 r_2 \cos\phi + r_2^2 \ ,$$

so that $\partial\alpha/\partial\delta$ becomes

$$\frac{\partial\alpha}{\partial\delta} = -\frac{2}{\pi} \left(\frac{r_2}{r_1}\right) \frac{\sin\phi}{\delta} \ . \tag{4.44}$$

Assembling all of these results allows A_{2m} to be written in the form

$$A_{2m} = -L_1 \int_{\delta_2}^{\delta_1} \left(\frac{\delta^2 - \delta_o^2}{1 - \delta_o^2}\right)^m \frac{\partial\alpha}{\partial\delta} \, d\delta \ . \tag{4.45}$$

Using this expression (for $m = 1,2,3$), and the previous

result for $\partial\alpha/\partial\delta$, one can generate equations for the A_{2m}'s. For $m=1,2,3$, one has (with $L_1 = 1-\lambda$)

$$\left.\begin{aligned} A_2 &= L_1 C_3 \\ A_4 &= L_1 (C_3^2 + C_2^2) \\ A_6 &= L_1 (C_3^3 + 3C_2^2 C_3 + C_1 C_2^2) \, , \end{aligned}\right\} \tag{4.46a}$$

where

$$\left.\begin{aligned} C_1 &= r_1^2 \csc^2 i \\ C_2 &= r_1 r_2 \csc^2 i \\ C_3 &= r_2^2 \csc^2 i - \cot^2 i \quad . \end{aligned}\right\} \tag{4.46b}$$

Solving this system of equations for r_1^2, r_2^2 and $\sin^2 i$ gives the following result:

$$r_{1,2}^2 = \frac{C_{1,2}^2}{(1-C_3)C_1 + C_2^2} \quad \text{and} \quad \sin^2 i = \frac{C_1}{(1-C_3)C_1 + C_2^2} \, . \tag{4.47}$$

Thus, we have expressions for the geometric elements of an eclipsing binary in closed form. However, the conditions for such a solution are that the disks of the stars appear uniformly bright (not too realistic), and that the eclipse be total. The results just obtained can be made to apply to an annular eclipse if r_1 and r_2 are interchanged, and if L_1 is replaced by $(r_2/r_1)^2 L_1$. The solution for a partial eclipse is an iterative one, and will not be discussed here. The procedure to be used is outlined by Kopal (1979, pg.155). The method of deriving the various results in this section

is indicative of the method used by Kopal in the more general cases of arbitrary limb darkening and non-spherical stars.

4.8 Total and Annular Eclipses of Limb-Darkened Stars

In most cases, the stars constituting an eclipsing binary system have a non-zero limb darkening. To modify the results obtained in the uniformly bright case to the limb darkened case amounts to little more than redefining the constants C_1, C_2, and C_3. The method by which this transformation is achieved, and the resulting method of solution for the elements will be outlined.

To pass to the case of non-zero limb darkening, a generalized form for the A_{2m}'s is required. Without going into the details of the derivation, the form of the A_{2m}'s derived by Kopal (1979, pg. 160) is

$$A_{2m} = \Gamma(m+1)\, L_1 \sin^{2m}\theta_1 \sum_{\ell=0}^{\infty} \frac{C^{(\ell)}}{\nu} \frac{(1-c_o^2)^{\nu+1}}{\Gamma(\nu+1)\Gamma(\nu+m+1)}$$
$$\times \sum_{n=0}^{\infty} \frac{\nu+2n+2}{n+1}(n+\nu+1)\left[\frac{\Gamma(\nu+n+1)}{\Gamma(n+1)} G_{n+1}(\nu,\nu+1,a)\right]^2$$
$$G_n(\nu+2,\, \nu+m+2,\, 1-c_o^2)\ , \qquad (4.48)$$

where $\Gamma(x)$ is the gamma-function, G_n is the Jacobi polynomial of order n (defined as a special case of the hypergeometric series ${}_2F_1(a,b,c,x)$, see Mathews and Walker, pg. 194), $C^{(\ell)}$ the constant defined in equation (4.29), $\nu = (\ell+2)/2$, $c_o = \cos^2 i$, and $a = r_1/(r_1+r_2)$. Explicit expressions for the A_{2m}'s for each case (total, annular,

partial) follow from the equation above.

The transformation to an arbitrary limb darkening can be carried out with the aid of equations (4.29) and the following equations (Kopal, 1979, pg. 164)

$$\left.\begin{aligned}
\bar{C}_3 &= C_3 \sum_{\ell=0}^{\infty} 1!\ \frac{C^{(\ell)}}{\nu} \\
\bar{C}_2^2 &= C_2^2 \sum_{\ell=0}^{\infty} \frac{2!\ C^{(\ell)}}{\nu(\nu+1)} \\
\bar{C}_1\bar{C}_2^2 &= C_1 C_2^2 \sum_{\ell=0}^{\infty} \frac{3!\ C^{(\ell)}}{\nu(\nu+1)(\nu+2)}
\end{aligned}\right\} \tag{4.49}$$

where C_1, C_2, C_3 are the constants introduced in the discussion of the uniformly bright case. In most cases of practical interest, only a linear limb darkening coefficient is available. In this event, the above equations reduce to

$$\left.\begin{aligned}
\bar{C}_3 &= C_3 \\
\bar{C}_2^2 &= C_2^2 \left(\frac{15 - 7u_1}{5(3 - u_1)} \right) \\
\bar{C}_1\bar{C}_2^2 &= C_1 C_2^2 \left(\frac{3(35 - 19u_1)}{70(3 - u_1)} \right) \quad .
\end{aligned}\right\} \tag{4.50}$$

These equations provide a means for introducing limb darkening into the method of analysis. Let us now consider the analysis of eclipsing binary stars with arbitrary limb darkening. The stars constituting the binary system will still be regarded as being spherical.

Once again, the simplest case is that of the total eclipse. The equations required are similar to those used

in the uniformly bright case (Kopal, 1979, pg. 164):

$$\left.\begin{aligned} A_o &= L_1 = 1-\lambda \\ A_2 &= L_1\bar{C}_3 \\ A_4 &= L_1(\bar{C}_3^2 + \bar{C}_2^2) \\ A_6 &= L_1(\bar{C}_3^3 + 3\,\bar{C}_2^2\bar{C}_3 + \bar{C}_1\bar{C}_2^2). \end{aligned}\right\} \qquad (4.51)$$

The solution proceeds in exactly the same way as in the uniformly bright case, once the limb darkening has been introduced using the equations presented earlier. However, the problem is more complicated in the case of annular and partial eclipses. The method to be presented here comes from a later work by Kopal (1982a). The equations relating the moments of the light curve, the A_{2m}'s, to the elements r_1, r_2, and i are (for both annular and partial eclipses):

$$\left.\begin{aligned} A_o &= L_1\alpha_o \\ A_2 &= L_1[\bar{C}_3 + (1-\alpha_o)\cot^2 i] - L_1B_2\cot^2 i \\ A_4 &= L_1[\bar{C}_3^2 + \bar{C}_2^2 - (1-\alpha_o)\cot^4 i] + L_1B_4\cot^4 i \\ A_6 &= L_1[C_3^3 + 3\bar{C}_2^2\bar{C}_3 + \bar{C}_1\bar{C}_2^2 + (1-\alpha_o)\cot^6 i] - L_1B_6\cot^6 i \end{aligned}\right\} \qquad (4.52)$$

where α_o is the value of α at mid-eclipse, and the B_{2m} are given by:

$$B_{2m} = -\left(\frac{r_2 \cos i}{r_1^2}\right)^2 \sum_{n=0}^{N} C^{(n)} \left(1 - \frac{r_2^2}{r_1^2}\right)^{n/2}$$

$$\times \sum_{j=0}^{\infty} \left(\frac{1}{(j+1)!} - \frac{m!}{(j+m+1)!}\right) \left(-\frac{n}{2}\right)_{j+1} \left(\frac{r_1 \cos i}{r_1^2 - r_2^2}\right)^{2j}$$

$$\times G_j(\nu - 2j, 2; r_2^2/r_1^2) \quad , \tag{4.53}$$

where G_j is the Jacobi polynomial and

$$\left(-\frac{n}{2}\right)_{j+1} = \frac{\Gamma(-\frac{n}{2} + j + 1)}{\Gamma(-\frac{n}{2})} \quad .$$

Kopal points out (1982, pg. 136) that the terms containing the B_{2m} are so numerically small that they may be ignored altogether or treated only as small perturbations. As an example in the same paper, Kopal computed B_2, B_4, and B_6 for the light curve of Algol. The largest term was $B_2 \cot^4 i$, having a magnitude of approximately 5×10^{-4}. $B_4 \cot^4 i$ and $B_6 \cot^6 i$ are smaller than $B_2 \cot^2 i$ by factors of 100 and 10000 respectively. The method of solution to be used is dependent upon the value of α_o. In the case of an annular eclipse, $\alpha_o = k^2 Y$, where $k = r_1/r_2$, and where

$$k^2 Y \equiv \frac{\alpha_o''}{\alpha_o'} = \frac{\alpha_o(\text{annular eclipse})}{\alpha_o(\text{total eclipse})} \tag{4.54}$$

serves to define Y. To solve for the elements in the case of an annular eclipse, one proceeds in the following way. We begin by finding k^2 from

$$k^2 = \frac{1 - \lambda_b}{\lambda_a Y} \quad , \tag{4.55}$$

where $\lambda_a = \ell$ at maximum total eclipse and $\lambda_b = \ell$ at maximum transit eclipse. A good starting value for Y is 1. L_1 may then be found with the aid of

$$A_o = 1-\lambda_b = L_1\alpha_o'' = L_1 k^2 Y \ . \tag{4.56}$$

One then solves for $x = r_2^2 \csc^2 i$ and $y = \cot^2 i$ from

$$x = \frac{A_o^2 A_6 + 2A_2^3 - 3A_o A_2 A_4}{A_o A_4 - A_2^2} \left[\frac{A_o(1+f_2k^2) - L_1}{A_o^2(1+3f_2k^2+f_4k^4) - 3A_oL_1(1+f_2k^2) + 2L_1^2} \right] \tag{4.57}$$

and $$y = \frac{xL_1 - A_2}{A_o} \ .$$

In the equation for x, f_2 and f_4 are

$$f_2 = \frac{15-7u}{5(3-u)} \quad \text{and} \quad f_4 = \frac{3(35-19u)}{35(3-u)} \ . \tag{4.58}$$

These equations, along with the fact that $r_1 = kr_2$, provide a preliminary set of elements. The value of k is recalculated, hence a new value of Y, and the procedure is repeated. Only two or three iterations are required to provide a final set of "good" elements. The B_{2m}'s may be included if one so desires. However, it should be remembered that uncertainties in the elements and the observations may not justify the use of the B_{2m}'s. For a variety of reasons, a solution for the elements could be poorly determined, so the use of the B_{2m}'s would only compound the problem.

4.9 Partial Eclipses of Limb-Darkened Stars

The solution for a partial eclipse is similar to that for an annular eclipse. In the case of a partial eclipse, one usually starts with k and Y both equal to 1. The value of α_o may be found from either of the following equations:

$$\alpha_o' = 1 - \lambda_a + \frac{1 - \lambda_b}{k^2 Y}$$

$$\alpha_o'' = 1 - \lambda_b + (1-\lambda_a) k^2 Y \quad , \tag{4.59}$$

since, for $k = Y = 1$, $\alpha_o = \alpha_o' = \alpha_o''$. The value of L_1 is found from $A_o = L_1 \alpha_o$. One now evaluates x and y as in the annular case. To find a new value of k, one can solve for k from

$$A_4 = L_1 [x - 2y + f_2 k^2 x] x + A_o y^2 \quad , \tag{4.60}$$

and then repeat the whole procedure until k no longer changes significantly from one iteration to the next.

Hence we have a straightforward iterative procedure for computing the elements of an eclipsing binary with limb darkening, for both annular and partial eclipses. It should be noted here that the definition of k, the ratio of the radii, depends on the type of eclipse. For a transit eclipse, $k > 1$, while for an occultation, $k < 1$. This arises from the fact that k is defined as the radius of the eclipsed star divided by the radius of the eclipsing star. Such a definition is quite useful in distinguishing eclipse types in the case of a partially-eclipsing system. Three

computer programs for determining the elements of an eclipsing binary for each of the three cases discussed above may be found in Appendix 3.

4.10 Incorporating the Effects of Non-Sphericity

If the stars constituting an eclipsing binary are non-spherical, one may still use the method of analysis just discussed, but the A_{2m}'s will have to be found with the aid of equations (4.21)-(4.26) or equations (4.27)-(4.31), i.e., the effects due to non-sphericity, reflection (and possibly mass transfer) will have to be filtered out before the solution for the elements can proceed. Other than this extra step in the computation, there are no added difficulties in solving for the elements of a non-spherical eclipsing binary. However, one should keep in mind that the radii so obtained are in fact mean radii.

4.11 Conclusions

This concludes the presentation of Kopal's frequency domain technique. The method obviously has several advantages, not the least of which is the ease with which it may be automated. The entire solution can be done with a digital computer, eliminating the need for cumbersome tables which are required in Russell's method. Kopal's method is also quite compact in its formulation, making it easily comprehensible. It also has the advantage of being a modern method of analysis, since it incorporates all current

knowledge of stellar structure. Moreover, all relevant astrophysical information can be brought to bear in those cases in which one is confronted with an eclipsing binary having certain peculiarities. However, no method of light curve analysis is without its drawbacks, and Kopal's method is no exception. The most apparent of these is the need to assume a value for the limb darkening (u_1) before proceeding with the analysis. One may infer a value for the limb darkening by considering the spectral type of each star. Unfortunately, the limb darkening inevitably turns out to be the most poorly determined element, no matter what method of analysis is used. A second drawback is the presence of the B_{2m}'s, whose purpose, it seems, is to make life difficult for the astronomer. Indeed, one should only consider using the B_{2m}'s if the light curve is of very high quality (small error $\Delta\ell$), or if the solution for the elements is very well determined. In any other situation, the B_{2m}'s would not be worth using. Putting these criticisms aside, Kopal's frequency domain technique is indeed a step forward in light curve analysis.

CHAPTER 5

WOOD'S MODEL AND THE WINK PROGRAM

5.1 Introduction

The last method of light curve analysis to be considered is an example of a "synthesis" method. Before exploring Wood's model and the WINK program in detail, let us consider some of the general features of all synthesis methods. Put very simply, a synthesis method is a pattern recognition algorithm. It keeps on constructing light curves and systematically varying the model parameters until it achieves a match with the observed light curve. The algorithm synthesizes the light curve using a model of the eclipsing binary, hence the name. Computer programs using the light curve synthesis approach are usually large (typically 2000 lines of code) and quite complicated. Most of the currently available programs are the end products of several years of development.

From a more technical point of view, synthesis programs utilize either a triaxial ellipsoid or Roche surface geometry for the component stars. Wood's model uses the former. A typical synthesis program proceeds in two general steps, the calculation of the luminosity ℓ at all observed phases θ, and a parameter adjustment step to bring the model into better agreement with the observed light curve. This step is usually achieved with the aid of a differential corrector operating in roughly the same way as in Russell's

method. The stellar models are usually complex, incorporating the properties of the atmospheres of both stars to determine parameters such as surface temperature, surface gravity, and limb darkening. It is also possible to account for extended atmospheres, which are found in Wolf-Rayet binaries and systems that have red giant components (a well known example of the latter is V V Cephei). To handle particularly close eclipsing systems, a typical synthesis program would incorporate a subroutine to handle the reflection effect. There is no standard procedure for handling reflection, and the approach used in each computer program is different. When dealing with the reflection effect, one has to make a tradeoff between computing time and the accuracy of the reflection model. Reflection effect calculations can significantly increase the running time of a synthesis program. Clearly, synthesis methods make use of a computer's "number-crunching" ability. This allows one to use the complex stellar models mentioned earlier. The first computer programs using the synthesis approach appeared in the early 1970's. Every researcher had a different approach. The earliest synthesis programs were those devised by Wilson and Devinney (1971), Hill and Hutchings (1970) and Lucy (1968). In each case, the approach taken was approximately the same, namely to use a Roche geometry, and to compute the surface intensity I(u) at several thousand points on the surface of each star. These programs were closely followed by those of Wood (1972), Rucinski (1973), Mochnacki and

Doughty (1972), Nelson and Davis (1972), and Berthier (1975). Mochnacki and Doughty were able to simplify the synthesis procedure by using cylindrical coordinates rather than the usual spherical bipolar coordinates. Berthier's program used a library of over 4000 known light curves to find initial values for model parameters. The Nelson-Davis approach uses an unorthodox integration procedure for determining surface brightness. More recently, Hill (1979), Binnendijk (1977), and Budding and Najim (1980) have proposed synthesis methods, with various improvements such as faster integration procedures, and an increased use of analytical formulae for computing certain quantities. Binnendijk's program even allows the user to compute a theoretical radial velocity curve for use with spectroscopic data. A paper by Hutchings (1971) describes the computational aspects of a synthesis method in some detail. Hutchings also discusses the computation of line absorption and emission profiles for eclipsing binaries, as well as the treatment of extended or expanding stellar atmospheres. The particular model that will be discussed here is Wood's model, in part because it is one of the most widely used programs. The WINK and WINK8 computer programs, both devised by Wood, will be discussed.

5.2 Model Parameters

We begin the survey of Wood's model and the WINK program by describing the general properties of the model. The treatment given here follows that of Wood (1971, 1972).

The model takes into account rotational and tidal distortion, limb darkening, gravity brightening, and reflection. The three major simplifications employed are that the stars are triaxial ellipsoids, with the reflection effect being approximated, rather than being computed by more rigorous (and consequently more time-consuming) means, and that the stars rotate in their orbital plane with a period equal to the orbital period. The first of these assumptions provides for fast computation, and is valid except for very close systems since the Roche surface defines the extent of the outer atmosphere and not the photosphere. Wood claims that the approximation used to simulate the reflection effect (to be discussed later) is "very good". This is to be interpreted in the sense that the approximation agrees well with more rigorous computations (i.e., the work of Napier (1968)). From an astrophysical point of view, the reflection approximation is a grey atmosphere approximation. The assumption regarding rotation can be relaxed somewhat, since orbital skew and polar tilt are allowed. Orbital skew implies that the stars do not face one another along their major axes, but that the major axis of one of the stars either lags or leads periastron passage by an amount σ. Polar tilt allows for stars whose rotation axis is not parallel to the orbital pole (by an amount ι). In each case, the deviations must be constant.

The model is defined by three sets of parameters, namely the orbital, geometric, and photometric parameters.

The orbital parameters serve to specify the nature of the relative orbit of star B about star A. Star A is eclipsed during the primary eclipse (i.e., it is the primary component). More specifically, the orbital parameters are the period P, time of conjunction T_c (usually zero), the semimajor axis of the orbit R_o (usually one), eccentricity e, longitude of periastron ω, and inclination i (see fig. 14). The definition of the inclination remains unchanged from previous chapters. The geometric parameters are just the stellar semiaxes a, b, and c. Certain relationships that exist between the semiaxes will be explored later. The photometric parameters specify the apparent intensity distribution on each star. These parameters are the surface intensity $\bar{I}$, or the value of I_o in

$$I = I_o(1 - u + u \cos \gamma)$$

at time $T_Q = T_c + P/4$ (time of quadrature), the limb darkening coefficient u, gravity brightening coefficient v and reflection coefficient (or albedo) w.

As mentioned earlier, there are certain relationships between the stellar semiaxes. Wood replaces the semiaxes a, b, and c by a new set of quantities

$$\begin{aligned} a_A &= a R_o \\ a_B &= k_a a R_o \quad , \quad \text{i.e.,} \quad k_a = a_B/a_A \\ b_A &= \varepsilon_A a R_o = \varepsilon_A a_A \\ b_B &= \varepsilon_B k_a a R_o = \varepsilon_B a_B \end{aligned} \tag{5.1}$$

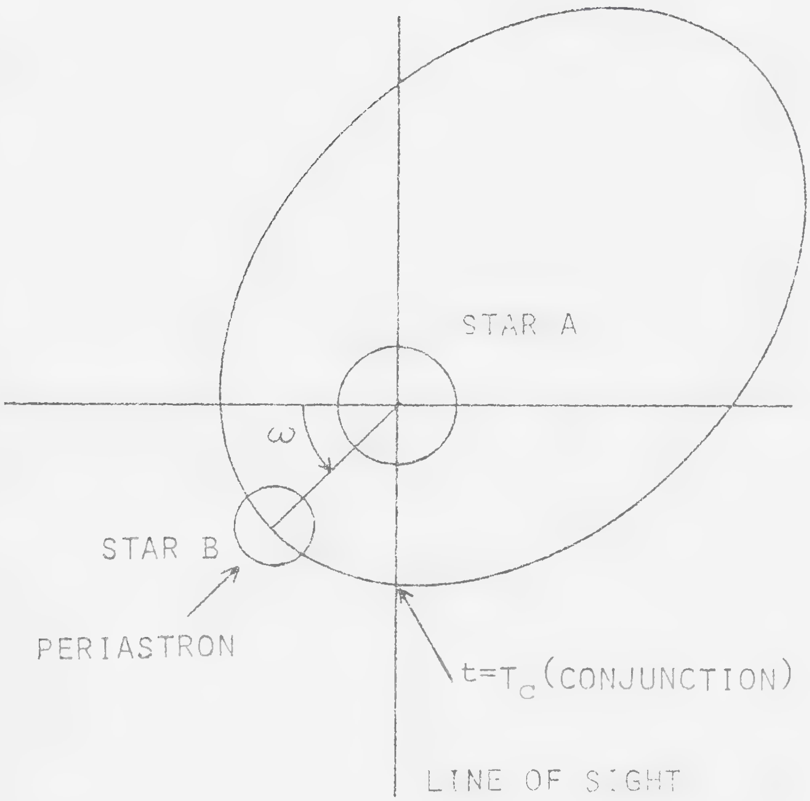


FIGURE 14. ORBITAL PARAMETERS IN WOOD'S MODEL.

$$c_A = (1+\zeta_A)\,\varepsilon_A^2\,a\,R_o = (1+\zeta_A)\,\varepsilon_A\,b_A$$

and $$c_B = (1+\zeta_B)\,\varepsilon_B^2\,k_a\,a\,R_o = (1+\zeta_B)\,\varepsilon_B\,b_B \quad .$$

The parameter k_a is so defined that if $k_a > 1$, then the primary eclipse is an occultation, and a transit if $k_a < 1$. The stars are assumed to be triaxial ellipsoids. Chandrasekhar (1933) showed that the stellar semiaxes can be written in terms of the polytropic index n, the mass ratio q (mass of star B/mass of star A), and a quantity $\nu = a_o/R_o$, where a_o is the "unperturbed radius" of the star, i.e., the radius of a sphere having a volume equal to that of the star. If each of the semiaxes is expanded in a series in ν, n, and q, one obtains the triaxial ellipsoid geometry (retaining terms up to order 3 in ν). It is convenient to introduce a quantity k_ν, which is defined as

$$\nu_B = \frac{a_{oB}}{R_o} = \frac{a_{oA}\,k_\nu}{R_o} \quad .$$

One can now express either the semiaxes or the quantities introduced in equation (5.1) in terms of a_{oA}, k_ν, q and n.

As in the models discussed in previous chapters, the intensity at any point on the apparent disk of either star may be written as

$$I = I_o(1 - u + u\cos\gamma) \ , \tag{5.2}$$

where I_o is the intensity at the "sub-earth" point, or the point at which the line of sight is perpendicular to the stellar surface ($\gamma = 0$). In the original WINK program

(version published by Wood in 1972), Wood uses a blackbody approximation for I_o. Improvements on this will be discussed later. The gravity brightening v is defined in the following way (analogous to the definition of limb darkening)

$$I_o = \bar{I}[1 - v + v(r/\bar{r})] , \tag{5.3}$$

where r is the local radius (at any point), and $\bar{r}$ is the radius at the sub-earth point at time T_Q. $\bar{I}$ was defined earlier as a photometric parameter.

The quantity which one must compare with the observational data is the system luminosity at a given time t. To compute this, one must evaluate

$$L = \iint (I_o + I^*) (1 - u + u \cos \gamma) \, dA \quad , \tag{5.4}$$

the integration being taken over the apparent ellipse of either star (shape of star projected onto the plane of the sky). The system luminosity at any time t is

$$L_{tot}(t) = L_A(t) + L_B(t) - L_{ecl}(t) \quad , \tag{5.5}$$

where L_{ecl} is the light loss during eclipse. L_A and L_B are so defined that at $t = T_Q$, $L_{tot} = L_A + L_B = 1$. The WINK program allows for the presence of a third star (the "third light") in the following sense:

$$L_A + L_B + L_C = 1 \; .$$

Hence, the calculation of the light curve amounts to the computation of $L_{tot}(t)$. In general, three integrations are

required, one over each star, and one over the overlapping area (if any). One must know the outline of each star and the overlapped area, as well as the intensity at any point on the stars. Also, reflection must be taken into account. The contribution to I from the reflection effect is the quantity I^* that appears in equation (5.4). Wood computes I^* by determining the local incident intensity L^* at a given point and then reflecting a fraction w uniformly over the outgoing hemisphere:

$$I^* = \frac{w\,L^*}{2\pi(1 - \frac{u}{2})} \quad . \tag{5.6}$$

I^* must now be added to I_o. The method used to compute L^* will be discussed in detail later, but the approximation used is valid for a_A and a_B less than 0.5 (the usual range of interest). Wood also allows for the possibility of an extended atmosphere around either star, but this aspect of the model will not be discussed here.

5.3 Computation of the Elements

The solution for the elements is done by least-squares differential corrections. The light curve intensity I is assumed to be of the form $I = I(t, X_1, \ldots, X_N)$, where X_i are the elements (unknowns). We may then write

$$\begin{aligned} \Delta I = I_{obs} - I_{calc} &= \frac{\partial I}{\partial X_1}\,\Delta X_1 + \ldots + \frac{\partial I}{\partial X_N}\Delta X_N \\ &= \sum_{i=1}^{N} \frac{\partial I}{\partial X_i}\,\Delta X_i \quad , \end{aligned} \tag{5.7}$$

where ΔX_i is the differential correction to the estimate X_i. Each observation provides an equation of condition having the form of equation (5.7). The resulting system of equations is solved by the least-squares method. This may be turned into an iterative process which, one hopes, will converge to a solution. Convergence is not easy to attain since there are observational errors and interrelationships between the parameters. It is also difficult to know which parameters should be varied, and a lot of guess work is involved. In general, more than one computer run is required. The partial derivatives $\partial I/\partial X_i$ are computed numerically, and there exist certain short-cuts and simplifications in their computation.

5.4 Details of the WINK Program

We now consider some specific aspects of Wood's model and the WINK program. Of greatest interest are the method of integration used, the way in which eclipses are detected and their limits found, the reflection approximation, and the use of model stellar atmospheres.

5.5 The Integration Procedure

In order to discuss the integration procedure used in the WINK program, we must first define the coordinate systems in which the integration is carried out. The fundamental coordinate system is a rectangular one centered on star A. The x-axis points along the line of sight and the

yz plane is in the plane of the sky. Each star has its own coordinate system, denoted by (x',y',z'). The z'-axis is coincident with the star's principle axis. Since there is octant symmetry, the sense of the coordinates does not matter (except for reflection effect calculations). The apparent ellipsoids each have a $\overline{y}\overline{z}$ coordinate system lying along the major axes of the apparent ellipsoids. The $\overline{y}\overline{z}$ system is rotated with respect to the yz plane.

The various integration procedures used in WINK are handled by the subroutines TOTINT (total eclipse), ANNECL (annular eclipse), ECLINT (partial eclipse) and ATMECL (atmospheric eclipse; replaces TOTINT). Subroutine TOTINT also computes the total light output from each star at any time. In all integration computations, Gaussian quadrature is used, with three grid sizes available: 4×4 (coarse), 6×6 (normal or default), and 12×12 (high precision). To integrate over an ellipse with semiaxes a and b, one uses the following formula:

$$I = a\,b \sum_{j=1}^{n} W_j \sqrt{1 - X_j^2} \sum_{i=1}^{n} W_i\, I_p\left(aX_j\,,\; X_i b\sqrt{1 - X_i^2}\right)$$

where $I_p(y,z)$ is the intensity along the line of sight at point (y,z) (equation (5.2)), W_i and W_j are the Gaussian weights, and X_i , X_j are the Gaussian ordinates (loaded by subroutine GRID). Also, n is the number of quadrature points (4, 6, or 12). Integration over an eclipsed area is done with the aid of

$$I = Y_D \sum_{j=1}^{n} W_j \, Z_D \sum_{i=1}^{n} W_i \, I_p(Y_D X_j + Y_S \, , \, Z_D X_i + Z_S)$$

where

$$Y_D = \frac{1}{2}\,(Y_H - Y_L) \quad , \quad Z_D = \frac{1}{2}\,(Z_H - Z_L)$$

$$Y_S = \frac{1}{2}\,(Y_H + Y_L) \quad , \quad Z_S = \frac{1}{2}\,(Z_H + Z_S) \; .$$

Y_H and Y_L are the y-limits of integration ($Y_H > Y_L$), found by subroutine LIMITY. Z_H and Z_L are the z-limits as found by subroutine LIMITZ, and are functions of $Y_j = Y_D X_j + Y_S$. In all cases, the integration is performed in the $\bar{y}\bar{z}$ coordinates, except for ECLINT, which uses a coordinate rotation. A coordinate translation is required in ANNECL because the integration is performed over the area of one star with the intensity points of the other star. A similar situation exists when star B is eclipsed, since the integration limits are in the coordinate system of star A. The intensity $I_p(y,z)$ is computed by the function BRIGHT. As indicated, BRIGHT uses the yz coordinates. Figure 15 shows the various integration grids.

5.6 Eclipse Detection and Limit Finding

It is the duty of the subroutines SCREEN, LIMITY, and LIMITZ to search for and find the limits of any eclipses that occur during computation of the light curve. Subroutine SCREEN searches for eclipses by comparing the centre-to-centre separation δ with the sum of the radius vectors.

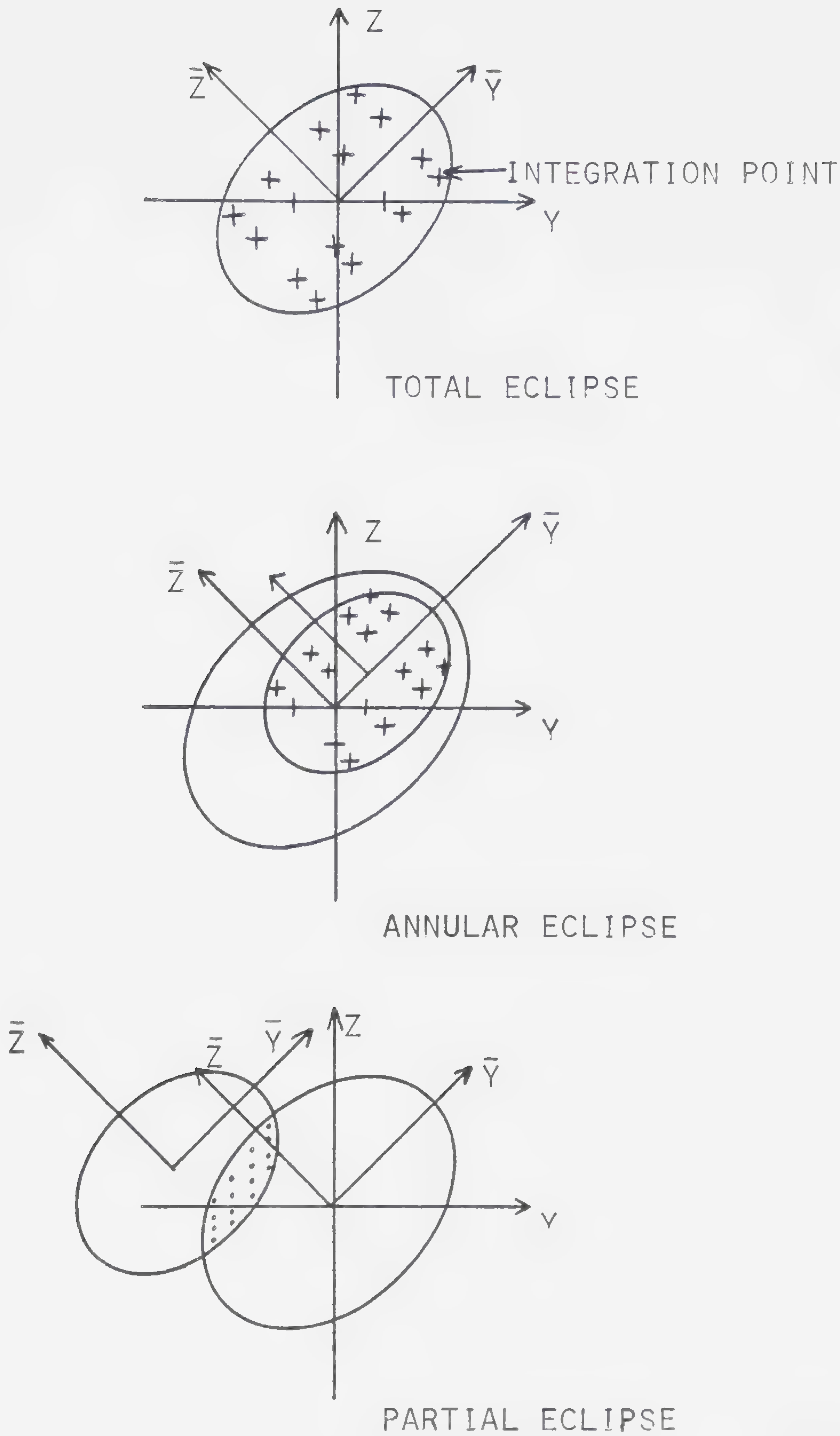
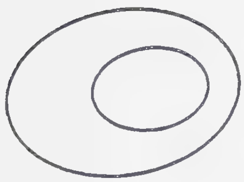


FIGURE 15. INTEGRATION GRIDS USED IN WINK (ADAPTED FROM WOOD(1972, PG.25)).

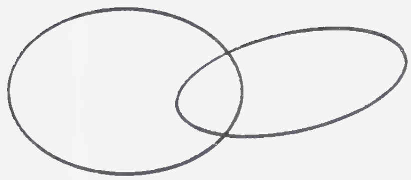
Obviously, an eclipse occurs if $\delta < r_1 + r_2$. Subroutines LIMITY and LIMITZ determine the y and z limits of the overlapping area during an eclipse and the nature of the eclipse (i.e., total, annular, or partial). In general, two ellipses can intersect at as many as four points. LIMITY (and LIMITZ) uses a bisection search (a line $y = \text{constant}$) to find these intersections. For an eclipse to occur, each ellipse must have two roots (four roots in total). The way in which the roots interleave determines whether an eclipse occurs, and if it is partial. The roots are denoted by Z_{A1} and Z_{A2} for star A, and by Z_{B1} and Z_{B2} for star B. An eclipse does not occur if Z_{A1} and Z_{A2} are both greater than, or both less than Z_{B1} and Z_{B2}. Figure 16 shows the various cases that can occur if an eclipse is suspected. The bisection search (Wood calls this a "scan wire") proceeds from right to left to determine where the interleaving of the roots changes. When such a change is found, the search is reversed and a smaller search interval is used. If the procedure is repeated, the location of the intersection may be determined quite accurately. In the event of a shallow eclipse, a smaller search interval is used, and if such an eclipse occurs on the negative y-axis, the negative limit is found first, or in other words, the search goes from left to right. The bisection algorithm is demonstrated in figure 17. A similar procedure is used to determine the z-limits of the eclipse, and it should be noted that the z-limits are usually the two "inner" roots (see figure 16).

2 ROOTS - NO ECLIPSE

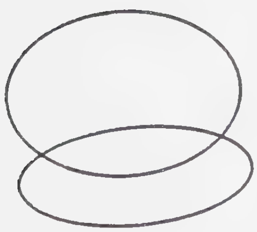
4 ROOTS - NO ECLIPSE



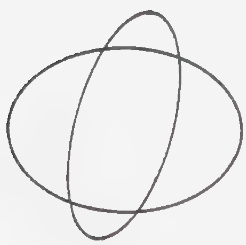
4 ROOTS - TOTAL OR ANNULAR ECLIPSE



4 ROOTS - PARTIAL ECLIPSE, STAR LIMBS AS INTEGRATION LIMITS.



4 ROOTS - PARTIAL ECLIPSE, INTERSECTIONS AS INTEGRATION LIMITS.



4 ROOTS - PARTIAL ECLIPSE, GENERAL CASE OF FOUR INTERSECTIONS.

FIGURE 16. IDENTIFICATION OF ECLIPSE TYPES (ADAPTED FROM WOOD(1972, PG. 21)).

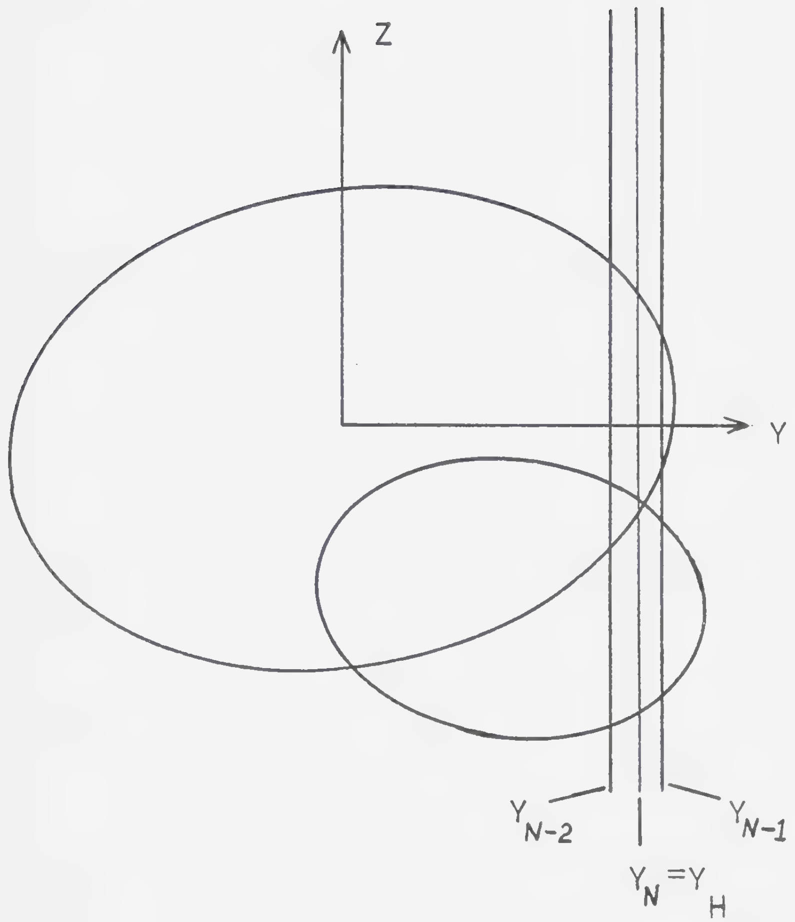


FIGURE 17. BISECTION SEARCH FOR DETECTING ECLIPSE LIMITS (ADAPTED FROM WOOD (1972, PG.22)).

5.7 The Reflection Effect

The reflection effect is handled by the subroutine REFL. The general procedure for determining the reflected intensity I^* was discussed earlier. Let us now examine the computation of L^*, the incident intensity. L^* is assumed to be a function of the source star's apparent angular size (A_s, as seen from a point on the other star), the intensity I_s at the end of its a-axis (i.e., the sub-stellar point), its limb darkening (u_s), and its apparent zenith distance (λ'). The approximation used was derived from more exact results obtained by numerical integration, since such an integration can increase the computing time required by an order of magnitude. We may now write L^* as

$$L^* = I_s A_s f(u_s) g(\cos \lambda') .$$

The limb darkening dependence, which appears through $f(u_s)$, is well approximated by

$$f(u_s) = 1 - \frac{u_s}{2}$$

since

$$\int_0^{\pi/2} (1 - u + u \cos \gamma) \sin\gamma \, d\gamma = 1 - \frac{u}{2} .$$

In other words, one is considering the contribution from the visible hemisphere of the source star. Wood has found that $g(\cos \lambda')$ can be approximated (linearly) as

$$g(\cos \lambda') = -0.065354 + a_r g_1 + (2.044 + a_r g_2) \cos \lambda' + C$$

where

$$g_1 = 0.224935 - 0.761696\, a_s + 3.81425\, a_s^2$$

$$g_2 = -0.170831 - 1.231707\, a_s + 9.955083\, a_s^2 ,$$

and C is the larger of 0 or

$$0.38736 + a_r(-0.82442 + 1.43431\, a_s) +$$
$$+ (\cos\lambda')[-1.22172 + a_r(-0.43316 + 4.9378\, a_s)] \quad .$$

In the above formulae, a_s and a_r are the apparent sizes of the source star and reflecting star respectively. The equations and procedure for determining λ' may be found in the publication by Wood (1972, Appendix 2).

5.8 Model Stellar Atmospheres

The original WINK program, as published by Wood in 1972, uses Planck's blackbody law to simulate the atmosphere of each star in an eclipsing binary system. In general, stars are not blackbody radiators, but over a narrow wavelength region, the approximation is valid. To allow U, B, and V light curves to be analyzed, one must have a better model of a star's atmosphere. To this end, fluxes computed from LTE (local thermodynamic equilibrium) model atmospheres were incorporated in succeeding versions of the WINK program. The current version, WINK8, uses the LTE model atmospheres computed by Kurucz (1979). By specifying the wavelength of observation (in Angströms), the effective temperature of each star (in degrees Kelvin), and the logarithm of the stellar surface gravity (g), WINK8

can interpolate in a table of fluxes to produce the required flux (and intensity) for the given wavelength, effective temperature, and log g. This procedure is handled by the function ATMOD. The atmosphere tables cover a range of surface gravities (log g = 2.5 to 4.5), wavelengths (3300 Å to 8000 Å), and two temperature ranges (4000 K to 9500 K and 11000 K to 40000 K). In the computation of the atmospheres, Kurucz used more realistic line opacities to simulate a wide variety of stellar spectral types. The atmospheres were also computed at 342 different wavelengths. Clearly, an approach such as this has a definite advantage over the blackbody approximation.

5.9 Conclusion

This brings us to the end of the discussion of Wood's model and the WINK program. Evidently, synthesis programs such as WINK and WINK8 use a much more realistic model of eclipsing binary systems, constituting a definite improvement over the Russell model. The stars are no longer restricted to a spherical geometry, so that one approach can be applied to a wide variety of systems. However, in certain cases, such as the W Ursae Majoris stars, one must resort to the Roche geometry. In contrast, Kopal's frequency domain method requires two different approaches, one for relatively "wide" binary systems, the other for close binary systems. Another advantage of a synthesis program is that complex or subtle effects can be handled with

relative ease (with the differential corrector), once the more fundamental parameters (such as radii and inclination) are known. Furthermore, the reflection effect is handled in a satisfactory manner, not only in a physical sense, but also in terms of computing time. Finally, realistic stellar atmospheres are used, allowing one to analyze the light curve at different wavelengths (i.e., BV or UBV). Model atmospheres such as these also allow a convective atmosphere to be simulated. Other atmospheric peculiarities such as starspots or extended atmospheres may also be included. The only drawback worth mentioning is that convergence to a solution is not guaranteed, and one must beware of non-global minima (i.e., $\sum(0-C)^2$ may have several minima, only one of which is the true solution (the deepest one)). However, good initial estimates of the elements and good observational data can alleviate most ambiguities. Clearly, models of eclipsing binary systems such as Wood's model provide one with a more detailed and physically realistic model, which is something of great necessity when one is dealing with eclipsing binary systems having photometric complications.

CHAPTER 6

SOME PRACTICAL GUIDELINES

From the preceding discussion, it is clear that some of the models of eclipsing binaries can only be used in certain situations, and that each requires a different amount of initial data. For an initial solution, the Russell-Merrill method, or the version due to Tabachnik, may be used. This will provide one with a set of "rough" elements, which may then be used as initial data for a synthesis program such as WINK8. Moreover, the Russell-Merrill method requires only one-half of an eclipse, and may therefore be used if the light curve is incomplete. The preceding remarks also apply to Kitamura's method, since it is a version of the Russell-Merrill method. Solutions made using either Kopal's frequency-domain technique or a synthesis method can be regarded as final, since both methods model a binary system in some detail. Kopal's method requires only one-half of an eclipse, but if there is a significant out-of-eclipse effect, this portion of the light curve is required. Since no initial estimates of the elements are required, one may make a solution directly from the data in one step. On the other hand, a synthesis program requires a complete light curve (implying the need for over one hundred data points) and an initial set of elements. However, the corresponding solution is (in most cases) of

high quality. Therefore, Kopal's method and the synthesis programs should be regarded as competitors. At this point, it is worth describing the procedure to be used if one is analyzing a close eclipsing binary. In this case, one should avoid the Russell-Merrill method, or any of its variants since the underlying physical model is not at all realistic (e.g., the lines of constant brightness on the surface of an ellipsoid are not concentric, contrary to the assumption made in the Russell model). Moreover, such an analysis would require the use of rectification, which is not a valid procedure. If one is attempting to analyze the light curve of a close eclipsing binary, either Kopal's method or a synthesis method should be used. To provide initial data for the latter, one may use a Russell-Merrill method since any effects due to either stellar distortion or to matter streams are at a minimum during eclipse. In fact, WINK8 provides its own set of default elements if no initial ones are available. In using Kopal's method, one must use the procedure described for the analysis of close eclipsing binaries (equations (4.21) to (4.26), or (4.14) and (4.27) to (4.31) of chapter 4). During any light curve analysis, it is advisable to assume a constant value for the limb darkening, since a variation in the limb darkening can mimic a change in the radius of either star (see Kopal (1979), pg. 232). These comments should serve as a broad guideline in the analysis of any eclipsing binary, but one should also be careful to account for the individual peculiarities of each system.

CHAPTER 7

H S HERCULIS

7.1 Introduction

The first example of the analysis of a light curve of an eclipsing binary will be that of H S Herculis. The data to be used in this analysis come from the paper of Hall and Hubbard (1971). The results obtained in this chapter will be compared to those of Hall and Hubbard, who used the Russell-Merrill method (with rectification) to obtain the geometric elements. A light curve appears in figure 18.

The most noticeable feature of H S Herculis is that the primary (deeper) eclipse is annular, and that the secondary is total. An annular eclipse is identified by a rounded minimum, unlike the sharp minimum of a partial eclipse, or the flat minimum of a total eclipse. Therefore, the larger star is also the brighter of the two. In most cases, the situation is reversed, with the smaller star being brighter. Since the secondary (total) eclipse is well-defined, it may be used in determining the geometric elements. Another feature that can be detected after some inspection is the displacement of the secondary minimum toward the primary minimum by an amount $\Delta\theta \simeq 0.02$ revolutions. One would therefore suspect a non-zero eccentricity and significant apsidal motion. Hall and Hubbard propose a value $e = 0.033$ for the eccentricity and an apsidal period

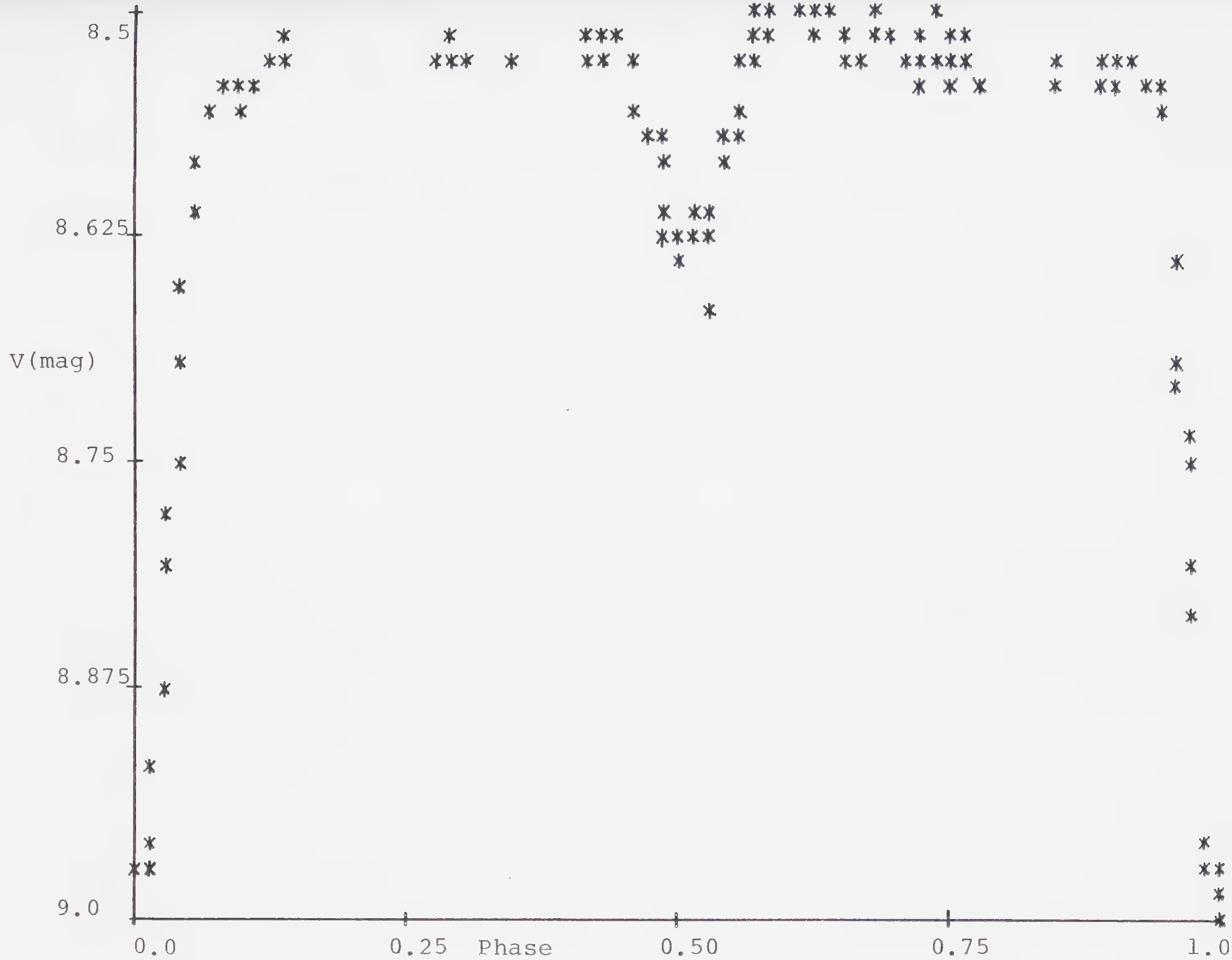


FIGURE 18. LIGHT CURVE-HS HERCULIS.

$P_A = 15.5$ yr. Scarfe and Barlow (1974) were unable to confirm this apsidal period. The values of e and P_A must be considered doubtful for another reason. The present author has discovered that the heliocentric Julian dates and phases as given by Hall and Hubbard do not coincide with one another (using Hall and Hubbard's values for P and t_o). There is a constant error of -0.0009 (±1 in the last place) revolutions when the phases of Hall and Hubbard are compared with those used in the present work. The cause of this discrepancy is not known, although the possibility of not making the heliocentric correction to the geocentric Julian dates has not been ruled out. The light curve shown in figure 18 incorporates the present author's phases. The two stars constituting the H S Herculis system are of early spectral type, with the larger one being of type B4 and the secondary of type A4, as given by Hall and Hubbard. The final feature worthy of note is that the secondary minimum is asymmetric, since the system is brighter after secondary eclipse than it is before the eclipse. This effect is most pronounced in the U light curve, but is barely noticeable in the V light curve. This asymmetry has been attributed to a gas stream near the inner Lagrangian point of the system, which is situated in such a way that it is invisible during the descending branch of the secondary eclipse, but reappears as the system comes out of eclipse.

7.2 Light Curve Analysis

The geometric elements, for the methods of Russell (Tabachnik's method), Kitamura, and Kopal are presented in table 7.1. The results from WINK8 are presented in table 7.2. In computing the geometric elements with the first three methods, the "depth equation" (equation (2.16)) was solved for k, the ratio of the radii. This was done using the iterative procedure suggested by Jurkevich (1970, pg. 75). In this procedure, one solves the depth equation for k^2:

$$k^2 = \frac{1 - \lambda_b}{\lambda_a Y(k - 1)} \qquad (7.1)$$

where

$$Y(k,-1) = [3(1 - x_b) + 2x_b({}^1\tau(k)/k^2]/(3 - x_b)$$

and

$${}^1\tau(k) = \frac{2}{3\pi}[3 \sin^{-1}\sqrt{k} - (3 - 4k)(1 + 2k)\sqrt{k(1 - k)}].$$

Here λ_b is the value of $\ell(\theta)$ at the moment of internal tangency during the transit eclipse and λ_a is the depth of the occultation eclipse. Also, x_b is the limb darkening of the larger star. Equation (7.1) is solved using a simple iterative procedure. A program for the TI-59 programmable calculator, which performs this calculation, may be found in Appendix 1. This procedure produced a value of $k = 0.582$ (for $x_b = 0.6$), which is slightly larger than Hall and Hubbard's 0.55. In all calculations, a limb darkening $x_b = 0.6$ was used.

Table 7.1 The Geometric Elements of H S Herculis

Russell Model (Tabachnik's method - total eclipse)

	initial	corrected	corrections (ZAPP.DC)
r_1	0.210	0.211	$6.01 \times 10^{-4} \pm 1.54 \times 10^{-3}$
r_2	0.122	0.122	$-1.13 \times 10^{-5} \pm 8.91 \times 10^{-4}$
k	0.582	0.578	
i	84°.63	84°.57	$\Delta(\cos^2 i) = 1.70 \times 10^{-4} \pm 4.32 \times 10^{-4}$
x	0.6	0.6	
L_1	0.900	0.902	0.002
L_2	0.100	0.098	$-1.98 \times 10^{-3} \pm 1.82 \times 10^{-4}$

Ratio of surface mean intensities $= \frac{J_a}{J_b} = \frac{L_2}{L_1} \frac{1}{k^2} = 0.325.$

Sum of squares of residuals $= \sum (0-C)^2 = 3.71 \times 10^{-7}$.

Russell Model (Kitamura's method - annular eclipse)

	initial	corrected	corrections (ZAPP.DC)
r_1	0.26	0.26	$0.1697 \times 10^{-3} \pm 0.4064 \times 10^{-2}$
r_2	0.16	0.16	$-0.1877 \times 10^{-3} \pm 0.5657 \times 10^{-2}$
k	0.62	0.62	
i	90°.0	89°.1	$\Delta(\cos^2 i) = -0.7095 \times 10^{-5} \pm 0.3735 \times 10^{-2}$
x	0.60	0.60	
L_1	0.900	0.944	0.0439 ± 0.0241
L_2	0.100	0.098	0.0439 ± 0.0241

Sum of squares of residuals $= \sum (0-C)^2 = 9.02 \times 10^{-6}$.

Table 7.1 (cont'd)

Kopal Model (annular eclipse)

r_1	(eclipsed star)	0.301 ± 0.006
r_2	(eclipsing star)	0.165 ± 0.034
k		1.82 ± 0.35
i		$82^\circ.12 \pm 2^\circ.14$
x		0.60 (fixed)
L_1	(eclipsed star)	0.902 (fixed)
L_2	(eclipsing star)	0.098 (fixed)

Moments of the light curve (program EB.FS):

$A_o = 0.3412 \pm 0.0017$

$A_2 = 0.01943 \pm 2.365 \times 10^{-4}$

$A_4 = 1.664 \times 10^{-3} \pm 5.599 \times 10^{-5}$

$A_6 = 1.792 \times 10^{-4} \pm 1.286 \times 10^{-5}$

Table 7.2 WINK8 results for H S Herculis

$r_1 = 0.261 \pm 0.003$

$k = 0.555 \pm 0.004$

$r_2 = 0.145 \pm 0.003$

$i = 86° \pm 0°.9$

Magnitude at quadrature = 8.522 ± 0.002 mag.

$T_1 = 23000 \pm 600°K$

$T_2 = 10000°K$ (fixed)

$x = 0.6$ (fixed)

$\log g_1 = \log_2 = 4.0$ (fixed)

q = mass ratio = 0.3 (fixed)

r.m.s. error = ±0.013

$\sum(0-C)^2 = 0.0227$

Number of iterations = 4

Astrophysical Parameters:

star	ν	a_o	T_{eq}(°K)	T_{pole}(°K)	Polytrope
A	0.262	0.262	23072.25	23333.63	5.0
B	0.145	0.145	10000.00	10064.91	5.0

Model: $a = 0.266$ $k_a = 0.553$ $k_\nu = 0.553$ $J_{5480} = 0.22$

$J_A/J_B = 0.25$ $J_{norm} = 0.22$ $J_{bolo} = 0.035$

star	ellipticity	ζ	u_1=x	v	w	a	b	c
A	0.9921	-0.0036	0.6	-4.0	1.0	0.266	0.264	0.261
B	0.9851	0.0085	0.6	-4.0	1.0	0.147	0.145	0.144

Table 7.2 (cont'd)

Luminosities:

star	apparent	normalized	total (4π)	normalized (4π)
A	0.175	0.929	1.38×10^{14}	0.98955
B	0.0134	0.0711	1.45×10^{12}	0.01045
ratio	0.0765		0.0106	

Reflection effect:

star	unheated (°K)	heated (°K)
A	22885.96	22893.32
B	9845.18	13709.76

Tabachnik's method (program LINE2) produced slightly smaller radii and a lower inclination. The most probable cause of this is that rectification was not used, since out-of-eclipse effects (other than the gas stream mentioned earlier) are minimal. One would expect rectification to alter the sizes of the stars and the inclination by a small amount since rectification produces an equivalent "spherical" system. Kitamura's method produced initial elements very close to those found by Hall and Hubbard. A differential correction (using program ZAPP.DC) did not produce any drastic changes in the elements. Kopal's method (programs KAL, ANNULAR, and EB.FS) was used with data from the annular eclipse. The elements found by ANNULAR are close to those found using Tabachnik's method. A further analysis, using programs EB.FS and ERROR, established the approximate errors in the elements. Program EB.FS (see Jurkevich (1980)) fits a Fourier series to the eclipse data, thus determining the Fourier coefficients and their uncertainties. It then determines the moments and their errors (see table 7.1). Program ERROR then uses these uncertainties, and the known values of the elements to compute their errors (see Kopal (1982a), pp. 154-155). The WINK8 solution used 143 data points, evenly distributed throughout the light curve of H S Herculis. The elements obtained were identical to those found by Hall and Hubbard. It is interesting to note that only four iterations were required (a maximum of six was allowed), and that in the last iteration, the inclination

was deleted from the list of variables to be corrected. WINK8 will do this if a correction ΔX_i is lower than a certain preset limit. One may conclude that, in the case of H S Herculis, the solution was very well determined. Reflection effect calculations revealed a significant contribution from the primary (hotter star), i.e., the temperature of the secondary (cooler) star is significantly affected by radiation from the primary. However, the converse is not true. However, in the WINK8 run, one temperature was held constant ($T_2 = 10000$ K) and the other allowed to vary. This would influence the magnitude of the reflection effect, so the result should be treated with some caution. Also, the temperature of the primary obtained by WINK8 does not agree with the result of Hall and Hubbard (for a B4 star $T \sim 16000$ K).

7.3 Conclusions

In concluding the analysis of H S Herculis, there are a few points that should be discussed. The most notable feature of the various sets of elements obtained is that their values are similar, but not the same (in general). These differences arise primarily from the different models and computational procedures used. In the case of Tabachnik's method, one is fitting a line to the variables $\sin^2\theta$ and $(1 + kp)^2$, so that the choice of k has an effect on the elements obtained. Another factor influencing the solution is the scatter of the points in the $\sin^2\theta - (1 + kp)^2$

plane, since a least-squares line minimizes the square distance (i.e., $(0-C)^2$) between the line and the data points. This is important because the elements depend on the slope and intercept of the line. Since Kitamura's method involves the use of the Fourier transform of $\ell(\theta)$, a smoothing operation is involved, decreasing the influence of scatter of the data points. Since Kitamura's tables are quite comprehensive, a good first approximation to the elements may be obtained. With reference to both Tabachnik's and Kitamura's methods, one must remember that they are "indirect" methods, since the elements are obtained through the use of $\sin^2\theta$, $(1+kp)^2$, E, F_2, and F_1/F_2. Therefore, elements differing by small amounts should be expected. Furthermore, it should be kept in mind that the Russell model, as used in the methods just discussed, is a first approximation only. The model is almost entirely geometric in character. The only physical features used are the limb darkening and the relative luminosities of the stars constituting the binary system. The essential point is that a simple, approximate model will give approximate elements. Since Kopal's method uses an entirely different approach to the problem, one would expect some differences between results obtained with it and with Russell-type methods. In Kopal's method, the limb darkening enters the problem directly, so one has a more physically realistic model (one can easily include non-linear limb darkening). As well, one has the advantage of the Fourier transform

operation and a subsequent error analysis. In complete contrast to the preceding methods, WINK8 fits the whole light curve to a model, whereas the previous methods use only eclipse data. Moreover, WINK8 uses a much more realistic model, in which the stars are treated as ellipsoids, not spheres. It also allows for finer details such as reflection, atmospheres (hence temperatures), both linear and non-linear limb darkening, and gravity brightening. There are also other parameters which may either be varied or held constant, according to the needs of the user. From the results obtained with WINK8, one can see that the components of H S Herculis are spherical to a very good approximation. Therefore, when one is comparing elements obtained by various methods, it is important to understand the assumptions and procedures involved in a particular solution method.

The next point to be addressed concerns the identification of the "best" set of elements. Technically, one should choose that set which minimizes $\sum(0\text{-}C)^2$. However, we must remember that the methods of Tabachnik, Kitamura, and Kopal use only eclipse data, and that WINK8 uses the whole light curve, as mentioned earlier. One cannot then compare the other three methods to WINK8, since these methods assume a constant value for $\ell(\theta)$ outside eclipses (in the spherical approximation). The methods of Tabachnik and Kitamura appear to give the best fit to the eclipse data (by considering $\sum(0\text{-}C)^2$), while WINK8 provides the

"best fit" to the whole light curve. One might then ask which method provides a set of elements closest to the set obtained by WINK8. From this point of view, the methods of Kitamura and Kopal are the closest. It should be noted that in general, the inclination has a greater uncertainty with respect to the other elements. This can be seen in various sets of elements obtained. Therefore, one can only choose a "best" set of elements after an intercomparison of the various sets of elements, and after considering the value of $\sum(O-C)^2$.

The final point to be raised regards the Russell model and rectification. The results obtained with WINK8 suggest that there is a large reflection effect. Hence, one may ask whether or not the Russell model, as used in the methods of Tabachnik and Kitamura, is really valid. To answer this, it should be remembered that the result obtained with WINK8 involved holding the secondary temperature constant, and letting that of the primary vary. As was mentioned earlier, this would influence the magnitude of the reflection effect. One should, therefore, treat the WINK8 result with some caution. Also, the Russell model is a first approximation, and serves to provide a preliminary set of elements. During the eclipses, reflection is being ignored. Furthermore, to make use of rectification would not be correct, because one cannot reliably extend a fitted curve beyond the domain of the fit. The rectification constants (the values of A_o, A_1, A_2 in the series $\ell(\theta)$ =

$A_o + A_1 \cos\theta + A_2 \cos^2\theta$) are fitted to the out-of-eclipse variation, and the resulting Fourier series is applied to all data points, both inside and outside eclipses. From the point of view of obtaining a set of preliminary elements, the use of the Russell model is justified.

In general, it is not possible to establish a set of criteria for choosing the best elements. Therefore, a choice of a "best" set elements should be based on the observer's experience, and on any other available information regarding the particular binary system.

CHAPTER 8

W DELPHINI

8.1 Introduction

The second star to be analyzed with the various methods of light curve analysis is W Delphini. This star has served as a "standard example" in earlier work on light curve analysis (see Russell (1912b), Aitken (1935, p. 192), Irwin (1962, p. 597), Tabachnik and Shul'berg (1967), and Tsesevich (1971)). The light curve of W Delphini has been well-observed by Wendell (1909, 1914), and more recently by Walter (1970). The data to be used in this chapter are taken from both of these sources. In particular, the primary eclipse data is taken from Irwin (1962), who used W Delphini to demonstrate the Russell-Merrill method (Irwin's data is from Wendell's light curve).

W Delphini has been classified as an "Algol-type" (i.e., detached) system by Russell (1912b), although the results of Walter (1970) indicate the presence of mass transfer and a significant reflection effect. The orbital period of W Delphini is 4.8061 days. The primary eclipse is a very deep total eclipse (flat minimum), the magnitude change being $2^m.695$. The secondary (annular) eclipse is very shallow and almost non-existent. Irwin (1962, p.599) has computed the change in magnitude to be $0^m.039$. The primary eclipse of W Delphini is shown in figure 19. In table 8.1, the values of the geometric elements, as found

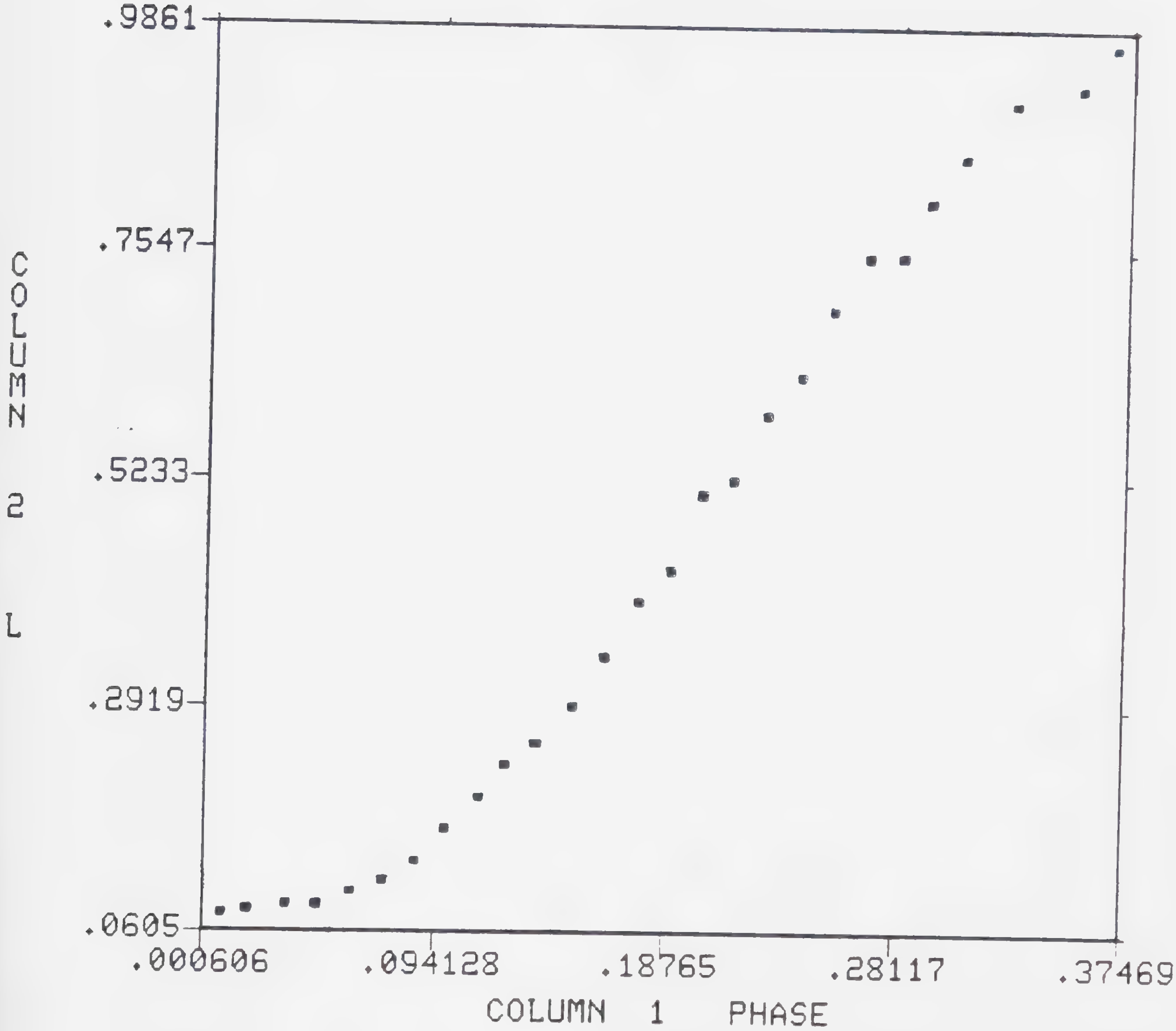


FIGURE 19. LIGHT CURVE-W DELPHINI.

by Irwin (1962, p. 598) are given. It should be noted that the value of $x = 0.5$ used for the limb darkening by Irwin was for demonstration purposes only. If the spectral types of the stars are taken into account, a more realistic value is $x = 0.6$. The spectral types, as given in the catalogue of Batten et al (1982) are A0e for the primary, and G5IV for the secondary. Furthermore, the eccentricity $e = 0.20$, the mass function $f(m) = 0.0127$, and $a \sin i = 1.94 \times 10^6$ km. The orbital elements of Irwin, as given in table 8.1, will be used as a comparison with results obtained by other methods.

8.2. Light Curve Analysis

The results obtained with the methods of Tabachnik, Kitamura and Kopal appear in table 8.2, and the WINK8 results in table 8.3. The light curve of Wendell was analyzed using the methods of Tabachnik, Kitamura, and Kopal, while Walter's light curve was analyzed with WINK8. (The observations were weighted according to the square root of the number of observations made at a particular phase). Tabachnik's method was programmed into a TI-59 programmable calculator. Once again, the program LCFT2 computed the quantities necessary for an analysis by Kitamura's method. An analysis using Kopal's method was carried out using the Kalman filter (program KAL) and the total eclipse program TOTAL. The Kalman filter was used because both light curves had relatively few points (Wendell: 27 points, Walter: 59

Table 8.1 W Delphini - Irwin's Elements

radius of large star = r_g = 0.2474 (k = 0.625)

radius of small star = r_s = 0.1546

$i = 85°.07$

light of large star = L_g = 0.9164

light of small star = L_s = 0.0836

limb darkening x = 0.6

Table 8.2 The Geometric Elements of W Delphini

Tabachnik's method (TI-59)

$i = 88°.04$

radius of large star = r_1 = 0.226

radius of small star = r_2 = 0.176

limb darkening x = 0.6

$L_1 = 0.9164$ $L_2 = 0.08360$

Kitamura's method (LCFT2)

$i = 88°$

$r_1 = 0.24$ $r_2 = 0.21$

$x = 0.6$

$L_1 = 0.9164$ $L_2 = 0.08360$

Kopal's method (KAL, TOTAL)

radius of eclipsed star = r_1 = 0.153

radius of eclipsing star = r_2 = 0.241

$i = 86°$

$L_1 = 0.9164$ $L_2 = 0.08360$

(Note: L_1 = light of larger star, L_2 = light of smaller star).

Table 8.3 WINK8 - W Delphini

$i = 85°$

radius of primary star = $r_1 = 0.147 \pm 0.006$

radius of secondary star = $r_2 = 0.241 \pm 0.022$

$L_A = 0.9213$

$L_B = 0.07868$ $\sum(0-C)^2 = 0.017573$

Astrophysical parameters

star	ν	a_o	β	T(equator)	T(polar)	log g	polytrope
A	0.14803	0.1480	0.250	9600.0 °K	9623.0 °K	4.0	5.0
B	0.24319	0.2432	0.250	4699.0°K	4796.0 °K	4.0	5.0

Model parameters

$a = 0.14863$ $k_a = 1.71916$ $k_\nu = 1.64284$

$J_{5310} = 0.02741$ $J_{norm} = 0.02723$ $J_{bolo} = 0.05742$

mass ratio = 0.5 quadrature magnitude = 0.025

star	ellipticity	ζ	u_1=x	u_2	v	w	a	b	c
A	0.09976	0.0	0.60	0.0	-4.0	1.0	0.1486	0.1483	0.1479
B	0.9589	0.0205	0.60	0.0	-4.0	1.0	0.2555	0.2450	0.2398

Luminosities

star	apparent	normalized	total (4π)	normalized (4π)
A	0.05538	0.92132	0.1299×10^{12}	0.86606
B	0.00473	0.07868	0.2009×10^{11}	0.13394
ratio	0.08540		0.15465	

points). In such a case, least-squares Fourier analysis would not produce meaningful results. This was substantiated when program EB.FS was used to compute the moments. Several trial runs produced situations in which some of the moments were negative, had large errors (greater than the moments themselves), or whose magnitude did not decrease with increasing values of m. On the other hand, the Kalman filter produced positive moments, whose magnitude did decrease with increasing values of m (see table 8.2). Unfortunately, the use of the Kalman filter precludes any error analysis. Moreover, the elements produced with the moments computed by KAL were much more reasonable. Other sets of moments (from EB.FS) produced values of the primary star's radius greater than one, which is an unreasonable value since the light curve is not one of an "over-contact" (i.e., similar to W Ursae Majoris) system. The analysis using WINK8 did not produce any unusual results, other than that it found both eclipses to be partial, rather than total and annular. However, such a result is not unreasonable, given that only 59 points were used, and that the eclipse is a grazing (i.e., just barely total) one. Hence, a small change in the inclination could alter the type of eclipse, from total to partial. As with HS Herculis, there is a significant reflection effect on the secondary star ($\sim 702\,°K$), but a negligible effect on the primary. The stellar semi-axes showed that both stars were spherical, with the secondary being slightly more polar-flattened than

the primary (see table 8.3). The complete analysis required five iterations.

8.3 Conclusions

In concluding the analysis of W Delphini, there are some points worth discussing. As with HS Herculis, there are differences between the different sets of elements. However, the differences are between the methods using the Russell model and those not using this model. The elements obtained with Tabachnik's and Kitamura's methods produced nearly identical results (except for the radius of the smaller star), while Kopal's method and WINK8 produced identical sets of elements. Once again, one is faced with the choice of choosing the best set of elements, but given the more realistic models employed by Kopal's method and WINK8, one would tend to believe the sets of elements obtained with these methods. It should be noted that a differential corrections program was not used, since the system is well-separated, and also for the purposes of a direct comparison between the methods. One can also argue that the elements obtained via Kopal's method are closest to those found by WINK8. Any further differences between the various sets of elements should be attributed to the computational methods employed.

The analysis just presented reveals an important point regarding Kopal's method. In the previous section, it was mentioned that the method of least-squares Fourier

analysis produced spurious results, making the use of the Kalman filter a necessity. This situation precluded any sort of error analysis. Hence, if one wishes to obtain a set of elements with error estimates using Kopal's method, one must have a large number of data points available. This situation also demonstrates the utility of the Kalman filter and its ability to produce good least-squares estimates of the moments using relatively few (27) points. Therefore, the Kalman filter is well-suited to situations in which the light curve consists of relatively few data points. The opposite would be true for least-squares Fourier analysis.

As a final note, this analysis of W Delphini shows that many of the eclipsing binaries analyzed using Russell-type methods would bear a second look, both from the point of view of light curve analysis, and from an observational point of view. Systems such as W Delphini should be reanalyzed using modern methods (Kopal or WINK8), which would use well-observed (i.e., several hundred data points) light curves as input data. One could then compute reliable sets of elements, with good error estimates. In this regard, this analysis of W Delphini is somewhat original since this system has not (to the author's knowledge) been analyzed with any method more complicated than Tabachnik's method. Rather than reaffirming old results, an analysis such as the one just presented provides greater insight into the phenomenon of eclipsing binaries.

CHAPTER 9

HD 219634

9.1 Introduction

The last eclipsing binary to be considered in this survey of light curve analysis methods is the recently discovered system of HD 219634. The system has been observed photometrically by Gulliver, Hube and Lowe (1982), but has been monitored only intermittently since the discovery of its eclipsing nature. Hube has also made several spectrographic observations, with which a spectroscopic solution has been computed (not yet published). The spectroscopic solution revealed a mass function of 0.16 solar masses and a period of 2.3912 days. The large value of the mass function suggested that HD 219634 might be an eclipsing binary. Gulliver et al have deduced a spectral type of BOVn for HD 219634. Since the secondary was not detected spectroscopically, this classification would refer to the primary star. It should also be noted that HD 219634 is a possible optical counterpart to the X-ray source 4U2316 + 61 (see Forman et al (1978)).

The partial light curve to be used in this analysis was obtained by Gulliver at Kitt Peak in 1981, using differential UBV photometry. The ΔV light curve appears in figure 20. The light curve has several noticeable features. The eclipses appear to be partial, since the minima are fairly sharp. Secondly, there is a noticeable lack of

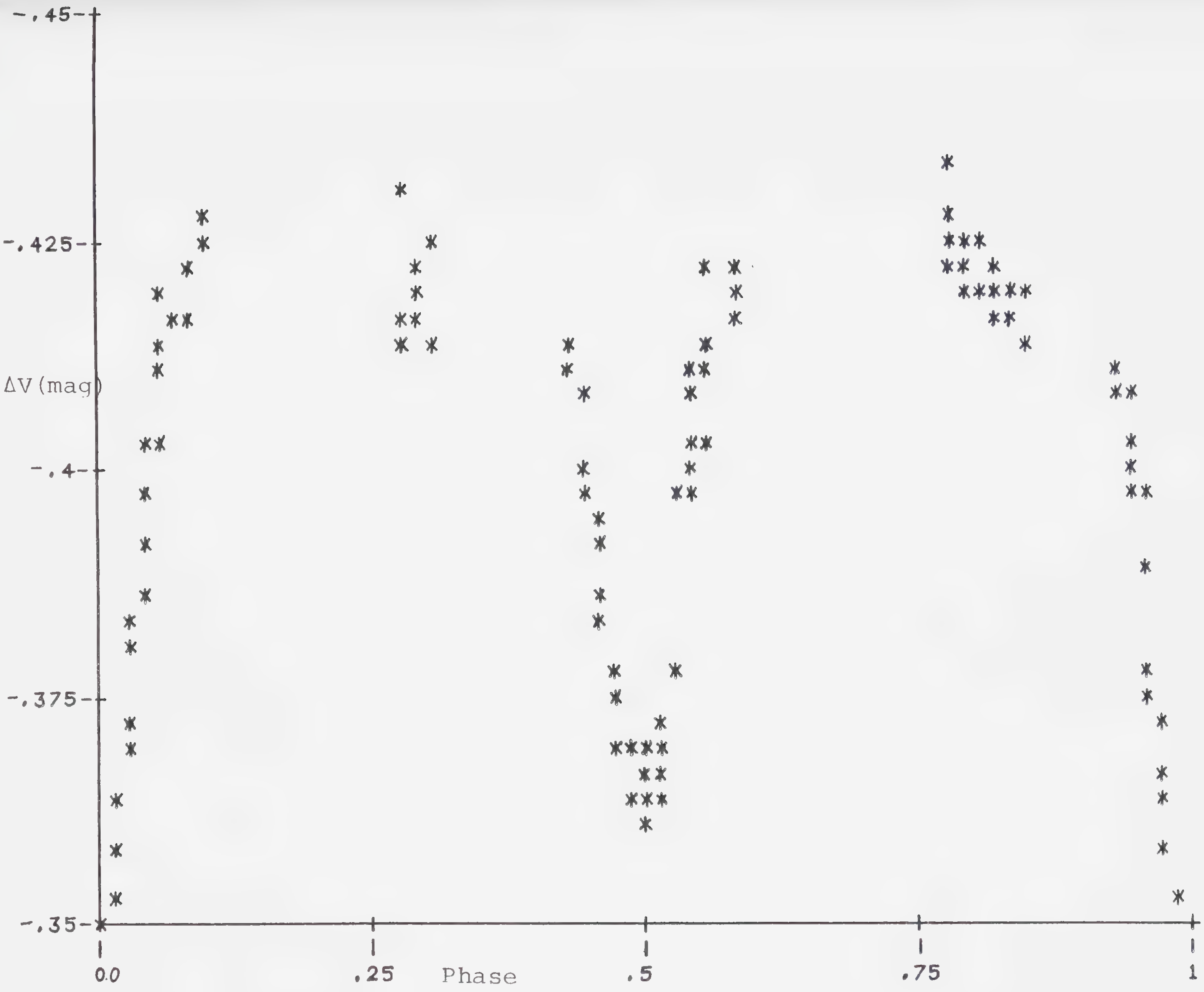


FIGURE 20. LIGHT CURVE-HD219634.

points outside the eclipses, a feature which automatically rules out least-squares Fourier analysis in Kopal's method. Hence, the Kalman filter must be used. The original light curve of Gulliver et al was asymmetric in the sense that the shoulders of the light curve (points of external tangency) were not level, particularly at primary minimum. The asymmetry amounted to $0^m.02$ in the ΔV light curve. The light curve shown in figure 19 has this variation removed (all points with $0.0 \leq \theta \leq 0.0930$ were shifted up by $0^m.02$). Gulliver et al have attributed this variation to a possibly non-constant comparison star (HD 220057) or to an intrinsic variation in HD 219634 itself. It should also be noted that the light variation outside the eclipses is not constant, so one would expect the components to be somewhat oblate. The observations to be used in the light curve analysis appear in appendix 5.

9.2 Light Curve Analysis

The two previous chapters have demonstrated the relative performance of the various methods of light curve analysis. HD 219634 allows these methods to be applied to a new, and previously unstudied system.

Since the light curve of HD 219634 shows effects due to oblateness and reflection, the Russell model is technically not applicable. However, the method of Kitamura was used since the Fourier transform operation minimizes irregularities in the light curve, and is therefore preferable

to Tabachnik's method. Kopal's method was used for the same reason. WINK8, which models both reflection and oblateness in a realistic manner, gave the best results. The results from the methods of Kopal and Kitamura appear in table 9.1. The WINK8 results appear in table 9.2. All methods produced an inclination close to 69° degrees. However, the agreement between the various radii is not good. In all cases, the radius of the smaller star lies between 0.1 and 0.2, while the larger radius lies between 0.12 and 0.36. A similar situation exists for the relative luminosities. Since WINK8 uses a more realistic model, the results so obtained are probably closer to the truth. The value of $k = 0.799$ obtained by WINK8 implies that the primary eclipse is a transit (i.e., annular). In using Kopal's method, it was found that the primary eclipse was an occultation. This was determined through the use of the Russell-Merrill χ-functions (see Appendix 3). No such assumption was required with Kitamura's method. The values of the characteristic function E indicated that the primary eclipse is an occultation, agreeing with Kopal's method. Further analysis to resolve this disagreement between the Fourier transform methods and WINK8 is most certainly required. Some other interesting results obtained with WINK8 were that the larger star is also the brighter one, and that there is a significant reflection effect on the secondary ($\Delta T \simeq 250°K$). Both stars are also significantly oblate. The temperatures obtained by WINK8 did not agree with the spectral classifi-

cation of Gulliver et al, which was BOVn (29000°K) for the primary and B8 (12000°K) for the secondary. This arises from the fact that WINK8 was allowed to use its default parameters in starting the solution. The default values of the temperatures are both 10000°K. This problem with the temperatures will be discussed further in the conclusions of this chapter. The discrepancies in the radii and relative luminosities displayed in the methods of Kopal and Kitamura can be attributed to the computational method used. In the case of Kopal's method, an initial estimate of k was required, and the resulting elements reflect the rather large uncertainty in the value of k. Unfortunately, the method for determining an improved value of k led to wildly erroneous values, so only an initial approximation to the elements was possible. In the case of Kitamura's method, it was possible to match the characteristic functions only for the case $r_b = 0.36$ and $r_a = 0.13$. Thus the problem lies with any uncertainties that the characteristic functions might have. However, the radii so obtained are closer to those found by WINK8 than are the radii from Kopal's method. Once again, similar remarks would apply to the relative luminosities.

9.3 Conclusions

In concluding this analysis of HD 219634, there are some points to be discussed and some observations to be made. In the previous section, it was mentioned that the

temperatures found by WINK8 did not agree with spectral classification of HD 219634. A possible remedy would be to fix both temperatures at the values determined from the spectral classification and to compute another solution. The temperatures would affect the solution through the limb darkening (which was held fixed at 0.6, according to the stars' spectral types) and through the value of the central intensity, I(0), since it is only these parameters which are directly affected by the temperature. However, one should not rule out more subtle connections between the temperatures of the stars and other parameters. Since a change in limb darkening can mimic a change in the radius, the limb darkening was held constant. Hence, the only direct effect that a change in temperature would have would be on I(0), which ultimately determines the luminosity of the star. This would obviously have an effect on the radius of the star, although the magnitude of such an effect could not be easily estimated. Hence, a further avenue of research would be to investigate the change in the radius induced by a change in the temperature. Clearly, the results obtained with Kopal's method require some comment. The problems lie in the computational method used. This method is outlined in Section 4.9 of Chapter 4. Using this approach, it seems to be quite difficult, if not impossible, to proceed to a more refined solution. The problem with the present method of solution in the case of partial eclipses is that an initial value of k must be specified. A better approach

would mean abandoning the method of Section 4.9 and to use a more fundamental approach. Kopal outlines such a method in his 1979 book (pp. 175-179). Without going into great detail, the method involves finding the roots of simultaneous non-linear equations

$$g_2(a,c_o) = \frac{A_2^2}{A_o A_4} \quad \text{and} \quad g_4(a,c_o) = \frac{A_4^2}{A_2 A_6}$$

where $$a = \frac{r_1}{r_1+r_2} \quad \text{and} \quad c_o = \frac{a\cos i}{r_1} \quad .$$

The right hand sides of the first set of equations are determined from the observations. The left hand sides are then computed using the series expansion, equation (4.48) of Chapter 4. The problem is then an iterative one for determining a and c_o.

Finally, the question arises regarding the best set of elements. It was mentioned in the previous section that the set of elements that are closest to the truth are those found by WINK8. Since WINK8 fits the whole light curve, and accounts for both oblateness and reflection, one would expect the WINK8 elements to be the best. The only other solution that comes anywhere close is the set of elements from Kitamura's method. With regard to WINK8, the problem of the discrepant temperatures should not be forgotten. Clearly, HD 219634 merits further investigation, both observationally and analytically. The number of out-of-eclipse observations is too small to reliably model the second-order effects, and the source(s) of the displacements of

the two branches of each eclipse is still not known. In spite of their small number and (possibly) large uncertainties, it is comforting to note that some sense can be made of the observations of this system.

Table 9.1 HD 219634 Orbital Elements

Kitamura

$i = 68°$

r_a (small star) = 0.12

r_b (large star) = 0.38

L_p (primary) = 0.08106

L_s (secondary) = 0.9189

$x = 0.6$

Kopal

$i = 68°.4$

r_1 (eclipsed star) = 0.168

r_2 (eclipsing star) = 0.122

L_1 (eclipsed star) = 0.2920

L_2 (eclipsing star) = 0.7080

$x = 0.6$

(Note: $r_1 = r_a$)

Table 9.2 HD 219634 Orbital Elements from WINK8

$i = 69°.06 \pm 0°.43$

magnitude at quadrature = -0.429 ± 0.001

$T_1 = 10000°K$ (fixed)

$T_2 = 9400°K \pm 251°K$

r_1(large star) = 0.249 ± 0.004

r_2(small star) = 0.199 ± 0.013

$k = 0.799 \pm 0.042$

$x = 0.6$ (fixed for both stars)

$\sum(0-C)^2 = 0.00793$

Astrophysical Parameters

star	ν	a_o	β	T(equator,°K)	T(polar,°K)	log g	poly-trope
A	0.25239	0.2524	0.25	10000.0	10155.81	4.0	5.0
B	0.18827	0.1883	0.25	9097.36	9139.15	4.0	5.0

Model Parameters

$a = 0.26054$ $k_a = 0.73230$ $k_\nu = 0.74597$ $J_{5200} = 0.78784$

$J_{norm} = 0.78873$ $J_{bolo} = 0.67955$ $q = 1.0$

star	ellipticity	ζ	$u_1(=x)$	u_2	v	w	a	b	c
A	0.9766	0.0076	0.6	0.0	-4.0	1.0	0.2605	0.2545	0.2504
B	0.9901	0.0033	0.6	0.0	-4.0	1.0	0.1908	0.1889	0.1877

Luminosities

star	apparent	normalized	total (4π)	normalized (4π)
A	0.16460	0.69642	0.4482×10^{12}	0.72697
B	0.07180	0.30358	0.1683×10^{12}	0.27303
ratio	0.43591		0.27557	

CHAPTER 10

CONCLUSIONS AND COMMENTS

After having applied the four methods of light curve analysis to three different eclipsing binaries, a number of questions quite naturally arise. To begin with, how do the various methods of light curve analysis compare? How physically realistic are the various models underlying these methods?

The analyses in the previous three chapters have clearly established that Wood's model is the "best" since the elements obtained are the result of a fit to the whole light curve, rather than to one half of an eclipse. Agreement with published results confirms that WINK8 is operating correctly. However, Kopal's frequency domain method performed just as well as WINK8, despite the fact that a spherical model was used. This is particularly evident in the analysis of W Delphini. One can therefore say that Kopal's method can produce results as good as those found using WINK8, but with relatively little computation. However, Kopal's method does not provide the detailed description that Wood's model does. For instance, one cannot estimate the magnitude of the reflection effect or compute the separate semiaxes a, b, and c in Kopal's method. Also, Kopal's method does not really allow one to compute a theoretical light curve, apart from computing α explicitly, a process which is quite tedious if the stars in a particular system

are significantly distorted. With these limits, however, one can regard the methods of Wood and Kopal as being equivalent. The two methods incorporating the Russell model, namely the methods of Tabachnik and Kitamura, show the approximate nature of the model, and hence its inability to represent systems which do not have spherical components, or those in which reflection may be important. Of these two methods, the more desirable of the two is that of Kitamura, since the Fourier transform operation acts as a smoothing process. The success of Kitamura's method in this regard is evident in the case of HD 219634, even though the elements were slightly different from those found by WINK8. A disadvantage that both of the Russell-type methods have is that they are indirect. One must compute various auxiliary functions and then either compare these with theoretical computations or use them in a further computation, which finally produces the set of elements. On the other hand, WINK8 and Kopal's method proceed to the solution more directly. In the case of WINK8, the residuals $\Delta I(0-C)$ have a direct bearing on the resultant elements. In Kopal's method, the moments of the light curve enter the equations for the elements directly.

As far as realism is concerned, the points made in the previous paragraph apply. The models of Wood and Kopal are clearly superior to the Russell model since the physical properties of the stars enter the analysis more directly. The Russell model is really little more than a geometric

one. With the models of Wood and Kopal, one can represent stellar oblateness and reflection rigorously, rather than resorting to the questionable approach of rectification. Also, the models of Wood and Kopal benefit from our current knowledge of stellar atmospheres, especially with regard to limb darkening. With regard to the reflection effect in particular, the models of Kopal and Wood handle it in a satisfactory way. WINK8 models the effect directly, whereas Kopal's method allows one to filter out the reflection effect.

The analyses of the previous chapters are but preliminary ones, particularly in the case of HD 219634. After such initial analyses, one can go on to refine the solution, investigating any interrelationships that may exist between the various model parameters. A case in point is the "discrepant temperatures" problem in HD 219634. Clearly, further work on this system is necessary. If the elements of a particular eclipsing binary are reasonably well determined, some other avenues of research are to determine the eccentricity of the relative orbit, the absolute dimensions of the system (using spectroscopic data), and to investigate the possibility of apsidal motion. The last item is particularly important with regard to relativistic effects and stellar evolution. Hence, one can gauge the importance of mass transfer. An independent approach would be to examine the spectrum of a system, which provides one with some idea of the rotational velocities of the two stars. There are

also other effects that can be investigated in the spectra of eclipsing binaries. The analyses of H S Herculis and W Delphini show that these systems are worth a second look, since the newer approaches to light curve analysis provide greater insight into the nature of eclipsing binaries, rather than reaffirming older results. Many eclipsing binaries analyzed with the Russell model would certainly benefit from more detailed consideration using one of the modern techniques.

In conclusion then, it can safely be said that modern methods of light curve analysis provide greater insight into the nature of eclipsing binary stars, since the models employed and the methods of analysis used are based on more realistic physical models. However, a "best set" of elements is not easy to obtain since light curve analysis methods are not "black boxes" which produce results independent of the user's judgement. Light curve analysis by its very nature is not a simple problem, so an "experimental" approach is required to really come to an understanding of the physics involved in eclipsing binary stars.

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Appendix 1 Russell Model Programs and Rectification

A1.1 Computer Programs

There are several programs which use Russell's model in the analysis of eclipse light curves. The FORTRAN programs LINE and LINE2 use Tabachnik's method. LINE takes a value of k and the values of θ and p as input data. After choosing an appropriate value of k, one must use the Russell-Merrill tables (see Russell and Merrill (1952)) to find the values of p corresponding to the values of α computed from the light curve. LINE does all other computations. LINE2 is similar to LINE, but takes as input values of p corresponding to two values of k (maximum difference of 0.1 in k allowed). LINE2 then scans all k values between the chosen limits, and prints out the corresponding values of $\sum(0-C)^2$. The value of k for which $\sum(0-C)^2$ is a minimum should be taken as the solution. Both LINE and LINE2 make use of an integer array known as INDIC. If INDIC(I) = 1, then a particular observation is included in the least-squares fit. An observation is omitted if INDIC(I) $\neq$ 1 (values for INDIC follow those of θ and p, i.e., θ_i, p_i, 1, for all observations). The programs are initiated by the commands

```
SET T = 1.0
$R *FORTG SCARDS = LINE  T=1.0
(Computer will respond with a message.)
$R  -LOAD#  5 = (A DATA FILE)  6 = -P
(another message.)
```

To see the results, list the output file -P. All FORTRAN programs are used in this fashion. A program using Tabachnik's method is also given for the TI-59 calculator. The instructions are self-explanatory. The programs ZAPP.DC and LIGHT are differential correction programs. ZAPP.DC is a WATFIV program that uses values of the derivatives from Irwin's tables (see Irwin (1947)). The generalized linear least-squares program used in ZAPP.DC has been adapted from the program published by Malin et al (1982). The values of the derivatives are found by interpolating in Irwin's tables. One need only enter the value of α, the values of the derivatives, the value of $\Delta\ell(0\text{-}C)$, the weight (usually 1.0), and the phase, for each observation. ZAPP.DC then formulates the equations of condition, and solves them for the corrections $\Delta L_{1,2}$, Δr_1, Δr_2, and $\Delta(\cos^2 i)$. The value of the limb darkening, which is required for the solution, is not corrected. The program listing includes a sample input. The standard deviations of the corrections are also printed out, as is the value of $\sum(0\text{-}C)^2$. A sample output follows the program listing. ZAPP.DC is initiated with the command

```
$R *WATFIV  SCARDS = ZAPP.DC .
```

All results are printed out on the terminal. Program LIGHT uses an entirely different approach. All derivatives are computed numerically, and the required α-functions are computed using the polynomial approximations of Fliegel and Wilson (1968). The least-squares procedure used is known

as the Levenberg-Marquardt algorithm (see, for example, Gill, Murray and Wright (1981, pp.136-137)). A program, known as CURFIT, which incorporates this algorithm has been published by Bevington (1969, pp. 237-239). This program, as well as other required subroutines have been incorporated into LIGHT. LIGHT requires only the input light curve (eclipse part only), as well as starting values for the elements and step lengths to be used in the solution. Due to its large size, LIGHT is not reproduced here, but a sample run and data file are given. LIGHT is initiated in the same way as LINE and LINE2, but one must use the following sequence to run the program:

$R - LOAD# 4 = COEF 5 = (A DATA FILE) 6 = -P

File COEF contains coefficients for the polynomial approximation to α. It should be noted that LIGHT cannot handle the annular phases of annular eclipses because an approximation to α for such phases does not yet exist. Such an approximation will be incorporated in future versions of LIGHT.

There are a few subtleties to be aware of when using LIGHT. One should not use "large" step lengths ($\sim$ 0.1) since this can produce wildly erroneous results. Hence, all step lengths should be no larger than 0.01, a good "typical" value being 0.001. In contrast to a differential corrector such as ZAPP.DC, LIGHT prints out the new set of elements followed by their standard deviations. Corrections are not printed out, but a list of input parameters is provided for the user's reference.

A1.2 An Example of Rectification

The method of rectification used is the graphical method of Russell and Merrill (1952, pg. 54), and is described completely therein. In order to rectify a light curve, one fits the Fourier series

$$\ell(\theta) = A_o + A_1 \cos\theta + A_2 \cos 2\theta + B_1 \sin\theta + B_2 \sin 2\theta$$

to the light variation outside eclipses. It should be noted that the presence of the last two terms cannot be justified physically. One is also assuming higher terms to be negligible, which is not usually the case. The method itself is quite simple: one reads values of $\ell(\theta)$ from the light curve at some convenient interval in θ (say $10° \simeq 0.17$ rad), letting a, b, c, d represent $\ell(\theta)$ at θ, $180°-\theta$, $180°+\theta$, and $360°-\theta$. By combining a, b, c, and d in various ways, one finds that

$$\frac{1}{4}(a+b+c+d) = A_o + A_2 \cos 2\theta$$

$$\frac{1}{4}(a-b-c+d) = A_1 \cos\theta$$

$$\frac{1}{4}(a+b-c-d) = B_1 \sin\theta$$

$$\frac{1}{4}(a-b+c-d) = B_2 \sin\theta \quad .$$

Therefore, by plotting the left hand side of these equations against $\cos 2\theta$, $\cos\theta$, $\sin\theta$, and $\sin 2\theta$, one can find the coefficients from the slopes (and intercept in the first equation only) of the lines. Any points happening to be within eclipse will not lie on any of the lines,

so one may effectively exclude in-eclipse observations, using this criterion. The rectification procedure is then carried out using equations (2.25) and (2.26) of chapter 2.

To illustrate the procedure, observations of H S Herculis outside eclipse will be analyzed. To facilitate the computation, only a small number of observations will be used. In general, it would be advisable to use all available observations. The procedure described above can be performed with relative ease using a programmable calculator. The coefficients A_o, A_1, A_2, B_1 and B_2 may then be found using a linear least-squares routine. The results of the calculation are presented in table A1.1, along with those of Hall and Hubbard (1971).

Table A1.1

	present work	Hall and Hubbard
A_o	0.9376	0.980
A_1	-0.01645	-0.015
A_2	-0.03931	0.003
B_1	0.0	0.0
B_2	0.0	0.0

```
C COMPUTATION OF ECLIPSING BINARY ORBITAL
C ELEMENTS BY TABACHNIK'S METHOD.
C PROGRAM 'LINE'. BY D.HOLMGREN, JAN.8'84.
C
      IMPLICIT REAL(A-H,O-Z)
      REAL INCL
      INTEGER INDIC(20)
      DIMENSION T(50),P(50),X(50),Y(50)
C READ IN DATA (THETA,P).
      READ(5,1)NOBS,RK
    1 FORMAT(I3,F14.7)
      READ(5,2) (T(I),P(I),INDIC(I), I=1,NOBS)
    2 FORMAT(2F14.7,I3)
C FORM X AND Y.
      DO 3 I=1,NOBS
      X(I)=SIN(T(I))*SIN(T(I))
    3 Y(I)=(1.+RK*P(I))*(1.+RK*P(I))
C DO LEAST-SQUARES FIT
      SY=0.
      SX=0.
      SY2=0.
      SX2=0.
      SXY=0.
      DO 4 J=1,NOBS
      IF(INDIC(J).NE.1) GO TO 4
      SX=SX+X(J)
      SY=SY+Y(J)
      SX2=SX2+X(J)*X(J)
      SY2=SY2+Y(J)*Y(J)
      SXY=SXY+X(J)*Y(J)
    4 CONTINUE
C DETERMINE PARAMETERS.
      AA=SXY-(SX*SY/NOBS)
      BB=SX2-SY2+((SY*SY-SX*SX)/NOBS)
      DSC=BB*BB/(4.*AA*AA)+1.
      IF(DSC .LT. 0.)DSC=-DSC
      A=-BB/(2.*AA)+SQRT(DSC)
      B=(SY-A*SX)/NOBS
      IF(A.LT.0.) A=-A
      IF(B .LT. 0.) B=-B
      TI=SQRT(A/B)
      INCL=ATAN(TI)
      R1=1./SQRT(A+B)
      R2=RK*R1
      WRITE(6,5)R1,R2,INCL
    5 FORMAT(1X,'R1=',F14.7,1X,'R2=',F14.7,1X,'INCL=',F14.7)
C FIND SUM SQ. RESID. .
      S2=0.
      DO 6 K=1,NOBS
    6 S2=S2+(A*X(K)-Y(K)+B)*(A*X(K)-Y(K)+B)
      S2=S2/(1.+A*A)
      WRITE(6,7)S2
    7 FORMAT('SUM SQ. RESID. =',F14.7)
      STOP
      END
END OF FILE
```

```
C COMPUTATION OF ECLIPSING BINARY ORBITAL
C ELEMENTS BY TABACHNIK'S METHOD.
C PROGRAM 'LINE2'. BY D.HOLMGREN, JAN.22'84.
C
      IMPLICIT REAL(A-H,O-Z)
      REAL INCL
      INTEGER INDIC(20)
      DIMENSION T(50),P(50),X(50),Y(50),PLO(50),PHI(50)
C READ IN DATA (THETA,P).
      READ(5,1)NOBS,NK,RKLO,RKHI,DEL
    1 FORMAT(2I3,3F14.7)
      READ(5,2) (T(I),PLO(I),PHI(I),INDIC(I), I=1,NOBS)
    2 FORMAT(3F14.7,I3)
C FORM X AND Y.
      RK=RKLO
      DO 100 L=1,NK
      CALL INTER(PLO,PHI,P,RK,RKLO,RKHI,NOBS)
      DO 3 I=1,NOBS
      X(I)=SIN(T(I))*SIN(T(I))
    3 Y(I)=(1.+RK*P(I))*(1.+RK*P(I))
C DO LEAST-SQUARES FIT
      SY=0.
      SX=0.
      SY2=0.
      SX2=0.
      SXY=0.
      DO 4 J=1,NOBS
      IF(INDIC(J).NE.1) GO TO 4
      SX=SX+X(J)
      SY=SY+Y(J)
      SX2=SX2+X(J)*X(J)
      SY2=SY2+Y(J)*Y(J)
      SXY=SXY+X(J)*Y(J)
    4 CONTINUE
C DETERMINE PARAMETERS.
      AA=SXY-(SX*SY/NOBS)
      BB=SX2-SY2+((SY*SY-SX*SX)/NOBS)
      DSC=BB*BB/(4.*AA*AA)+1.
      IF(DSC .LT. 0.)DSC=-DSC
      A=-BB/(2.*AA)+SQRT(DSC)
      B=(SY-A*SX)/NOBS
      IF(A.LT.0.) A=-A
      IF(B .LT. 0.) B=-B
      TI=SQRT(A/B)
      INCL=ATAN(TI)
      R1=1./SQRT(A+B)
      R2=RK*R1
      WRITE(6,5)R1,R2,INCL
    5 FORMAT(1X,'R1=',F14.7,1X,'R2=',F14.7,1X,'INCL=',F14.7)
      WRITE(6,12) RK
   12 FORMAT(1X,'K=',F14.7)
C FIND SUM SQ. RESID. .
      S2=0.
      DO 6 K=1,NOBS
    6 S2=S2+(A*X(K)-Y(K)+B)*(A*X(K)-Y(K)+B)
      S2=S2/(1.+A*A)
      WRITE(6,7)S2
    7 FORMAT('SUM SQ. RESID. =',F14.7/)
  100 RK=RK+DEL
      STOP
```

```
      END
C LINEAR INTERPOLATION SUBROUTINE.
      SUBROUTINE INTER(PLO,PHI,P,RK,RKLO,RKHI,NOBS)
      IMPLICIT REAL(A-H,O-Z)
      DIMENSION PLO(50),PHI(50),P(50)
      DO 1 I=1,NOBS
    1 P(I)=PLO(I)+(PHI(I)-PLO(I))*(RK-RLO)/(RKHI-RKLO)
      RETURN
      END
END OF FILE
```

```
12,.40,
2.7489,1.000,2,
2.7539,0.7749,1,
2.7696,0.6601,1,
2.8067,0.2482,1,
2.8255,0.1079,1,
2.8306,0.003974,1,
2.8463,-0.05165,1,
2.8727,-0.2541,1,
2.9506,-0.9401,1,
2.9965,-0.6795,1,
3.0065,-0.6427,1,
3.0190,-1.000,2,
END OF FILE
```

```
27,10,0.5,0.6,0.010,
0.0012,-1.000,-1.000,1,
0.0029,-1.000,-1.000,1,
0.0054,-0.9541,-0.9574,1,
0.0074,-0.9490,-0.9526,1,
0.0096,-0.8908,-0.8969,1,
0.0117,-0.8538,-0.8614,1,
0.0137,-0.7955,-0.8056,1,
0.0150,-0.7134,-0.7057,1,
0.0179,-0.6462,-0.6602,1,
0.0195,-0.5810,-0.5960,1,
0.0216,-0.5438,-0.5593,1,
0.0239,-0.4776,-0.4938,1,
0.0259,-0.3941,-0.4286,1,
0.0281,-0.2997,-0.3165,1,
0.0301,-0.2509,-0.2676,1,
0.0322,-0.1274,-0.1437,1,
0.0341,-0.1048,-0.1210,1,
0.0363,0.0016,-0.0137,1,
0.0384,0.0658,0.0511,1,
0.0404,0.1748,0.1650,1,
0.0427,0.2727,0.2603,1,
0.0449,0.2727,0.2603,1,
0.0466,0.3807,0.3701,1,
0.0488,0.4752,0.4659,1,
0.0522,0.6086,0.6012,1,
0.0563,0.6537,0.6473,1,
0.0585,0.7954,0.7913,1,
END OF FILE
```

TI PROGRAMMABLE

PROGRAM RECORD
FICHE PROGRAMME

TITLE / TITRE: Russell Model — PAGE 1 OF / DE 3

PROGRAMMER / PROGRAMMEUR: D.Holmgren — DATE: May 19,1983

PARTITIONING (OP 17) / PARTITION (OP 17): 479.59 — LIBRARY MODULE / MODULE ENFICHABLE: - — PRINTER / IMPRIMANTE: - — CARDS / CARTES: 1

PROGRAM DESCRIPTION • DESCRIPTION DU PROGRAMME

Program solves for the elements of an eclipsing binary (r_1,r_2,i) in the case of complete eclipses (ie., total or annular).
Method used is a variation of Kopal's method due to V.M. Tabachnik, which fits a line of the form y=ax+b, where $y=(1+kp)^2$, $x=\sin^2$, $a=\sin^2 i/r_1^2$ and $b=\cos^2 i/r_1^2$ ($k=r_2/r_1$). p is the geometrical depth, and is used as input data (use Russell-Merrill tables of $\alpha(k,p)$). Elements are given by: tan i=a/b,$r=(a+b)^{-1/2}$,$r_2=kr_1$ (k used as input)

USER INSTRUCTION • MODE D'EMPLOI

STEP SEQUENCE	PROCEDURE	ENTER INTRODUIRE	PRESS APPUYER SUR	DISPLAY AFFICHAGE
0	Load program		1	1
1	Initialize		E	0.
2	Input k	k	r/s	intermediate result
3	Input data (θ_j,p), where θ_j is in radians and p_j is the corresponding geometrical depth.	θ_j	A	intermediate result.
		p_j	r/s	number of points so far
4	Calculate elemnts ("GO")- first element computed is i.		B	i(degrees)
5	Display r_1		r/s	r_1
6	Display r_2		r/s	r_2
7	Examine correlation coefficient (CC) as a check on the value of k used (for k close to the correct value, the CC should be close to 1)		r/s	CC

USER DEFINED KEYS / TOUCHES UTILISATEUR	
A	data entry
B	compute
C	
D	
E	initialize
A'	
B'	
C'	
D'	
E'	

DATA REGISTERS / REGISTRES-MEMOIRE (INV List)

0	k	10	r_1
1–6	linear regression (L.R.)	11	r_2
7	b	12	
8	a	13	
9	L.R.	14–19	

LABELS (OP 08)

INV lnx CE CLR x:t x² √x 1/x STO RCL SUM y^x EE ( ) ÷ GTO × SBR − RST + R/S . +/− = CLR INV log CP tan Pgm P→R sin cos CMs Exc Prd |x| Eng Fix Int Deg Pause x=t Nop Op Rad Lbl x≥t Σ+ x̄ Grad St flg If flg D.MS π List Write Dsz Adv Prt

FLAGS / DRAPEAUX: 0 1 2 3 4 5 6 7 8 9

TITLE / TITRE: Russell Model PAGE 2 OF / DE 3

TI PROGRAMMABLE

PROGRAMMER / PROGRAMMEUR: D. Holmgren DATE: may 19,1983

CODING FORM

FEUILLE DE PROGRAMMATION

152a

LOC ADR	CODE	KEY TOUCHE	COMMENTS COMMENTAIRES
000	76	LBL	
001	15	E	
002	36	PGM	
003	01	01	
004	71	SBR	
005	25	CLR	
006	70	RAD	
007	91	R/S	
008	42	STO	
009	00	00	
010	91	R/S	
011	76	LBL	
012	11	A	
013	38	SIN	
014	33	X²	
015	32	X⇄T	
016	91	R/S	
017	65	×	
018	43	RCL	
019	00	00	
020	85	+	
021	01	1	
022	95	=	
023	33	X²	
024	78	Σ+	
025	91	R/S	
026	76	LBL	
027	12	B	
028	69	OP	
029	12	12	
030	42	STO	
031	07	07	
032	32	X⇄T	
033	42	STO	
034	08	08	
035	55	÷	
036	43	RCL	
037	07	07	
038	95	=	
039	34	√X	
040	22	INV	
041	30	TAN	
042	65	×	
043	01	1	
044	08	8	
045	00	0	
046	55	÷	
047	89	π	
048	95	=	
049	91	R/S	
050	43	RCL	
051	07	07	
052	85	+	
053	43	RCL	
054	08	08	
055	95	=	
056	34	√X	
057	35	1/X	
058	42	STO	
059	10	10	
060	91	R/S	
061	43	RCL	
062	10	10	
063	65	×	
064	43	RCL	
065	00	00	
066	95	=	
067	42	STO	
068	11	11	
069	91	R/S	
070	69	OP	
071	13	13	
072	91	R/S	
073	91	RST	

MERGED CODES
TOUCHES COMBINEES

62 | 72 STO | 83 GTO
63 | 73 RCL | 84
64 | 74 SUM | 92 INV SBR

TEXAS INSTRUMENTS

TITLE / TITRE: Russell Model — PAGE 3 OF/DE 3 — TI PROGRAMMABLE

PROGRAM RECORD / FICHE PROGRAMME

PROGRAMMER / PROGRAMMEUR: D. Holmgren — DATE: May 19,1983

PARTITIONING (OP 17) / PARTITION (OP 17): 47 9.59 — LIBRARY MODULE / MODULE ENFICHABLE: 1 — PRINTER / IMPRIMANTE: - — CARDS / CARTES: 1

PROGRAM DESCRIPTION • DESCRIPTION DU PROGRAMME

Example: W Delphini - total eclipse. k is estimated from:

$$k=(\theta'-\theta'')/(\theta'+\theta'') = 0.5779$$

where θ'=phase at first contact, θ''=phase at second contact. See sheet of observations which follows program description. Results:

k=0.5779:	i=84.45 (degrees)	k=0.528:	i=83.42
	r_1=0.248		r_1=0.256
	r_2=0.144		r_2=0.135 (published)

USER INSTRUCTION • MODE D'EMPLOI

STEP SEQUENCE	PROCEDURE	ENTER INTRODUIRE	PRESS APPUYER SUR	DISPLAY AFFICHAGE
0	Load program		1	1
1	Initialize		E	0.
2	Enter k	0.5779	r/s	0.5779
3	Enter data	θ_1=0.394	A	0.
		p_1=1.00	r/s	1.
		θ_2=0.3304	A	1.1473....
	etc., until all observations have been entered.	p_2=0.64	r/s	2.
4	Calculate i		B	84.45
5	Calculate r_1		r/s	0.248
6	Calculate r_2		r/s	0.144
7	Check CC		r/s	0.99974....
	Solution complete			
	Ref. Eclipsing Variable Stars, Ch.5, Edited by V.P. Tsesevich, John Wiley & Sons, 1973.			

USER DEFINED KEYS / TOUCHES UTILISATEUR: A, B, C, D, E, A', B', C', D', E'

DATA REGISTERS / REGISTRES-MEMOIRE (INV List): 0–9, 0–9

LABELS (OP 08): INV, lnx, CE, CLR, x:t, x², √x, 1/x, STO, RCL, SUM, yˣ, EE, (,), ÷, GTO, ×, SBR, −, RST, +, R/S, ., +/−, =, CLR, INV, log, CP, tan, Pgm, P→R, sin, cos, CMs, Exc, Prd, |x|, Eng, Fix, Int, Deg, Pause, x=t, Nop, Op, Rad, Lbl, x≥t, Σ+, x̄, Grad, St flg, If flg, D.MS, π, List, Write, Dsz, Adv, Prt

FLAGS / DRAPEAUX: 0 1 2 3 4 5 6 7 8 9

TITLE / TITRE: Depth Equation Solution. PAGE / DE: 1 OF / DE 3

TI PROGRAMMABLE

PROGRAM RECORD
FICHE PROGRAMME

PROGRAMMER / PROGRAMMEUR: D. Holmgren DATE: Jan. 27'84

PARTITIONING (OP 17) / PARTITION (OP 17): 4 7 95 9 LIBRARY MODULE / MODULE ENFICHABLE: - PRINTER / IMPRIMANTE: - CARDS / CARTES: 1

PROGRAM DESCRIPTION • DESCRIPTION DU PROGRAMME

Solves the depth equation using an iterative procedure. Input data is the limb darkening x (larger star), $1-\lambda_b$ ($\lambda=\ell(\theta)$ at internal tangency of annular eclipse), λ_a=depth of total eclipse (or value of $\ell(\theta)$ at internal tangency of total eclipse), and an initial estimate of k.

The equations are:

$$k^2=\frac{1-\lambda_b}{\lambda_a Y(k,-1)}$$

$$Y(k,-1)=(3(1-x_b)+2x_b(1_\tau(k)/k^2))/(3-x_b)$$

$$1_\tau(k)=\frac{2}{3\pi}\left(3\sin^{-1}\sqrt{k}-(3-4k)(1+2k)\sqrt{k(1-k)}\right)$$

USER INSTRUCTION • MODE D'EMPLOI

STEP SEQUENCE	PROCEDURE	ENTER INTRODUIRE	PRESS APPUYER SUR		DISPLAY AFFICHAGE
0	Load Program				
1	Enter input data (x_b, $1-\lambda_a$, λ_b, k)	x_b	STO	00	x_b
	(initial estimate of k)	$1-\lambda_b$	STO	01	$1-\lambda_b$
		λ_a	STO	02	λ_a
		k	STO	03	k
2	Set radian mode	-	2ND	RAD	
3	Begin iteration		r/s		[flashing
4	See new approximate k				new k
5	Continue iteration (as many times as necessary to obtain convergence)		r/s		
			r/s		final k
	*Requires only ~four iterations with a good initial estimate. A bad estimate will extend the process by 2 or 3 iterations.				

USER DEFINED KEYS / TOUCHES UTILISATEUR		DATA REGISTERS / REGISTRES-MEMOIRE (INV List)			
A	$1_\tau(k)$ routine	0	x_b	0	
B		1	$1-\lambda_b$	1	
C		2	λ_a	2	
D		3	k	3	
E		4	Y(k,-1)	4	
A'		5	$1_\tau(k)$	5	
B'		6		6	
C'		7		7	
D'		8		8	
E'		9		9	

LABELS (OP 08)

INV	lnx	CE	CLR	x⇌t	x^2
$\sqrt{x}$	1/x	STO	RCL	SUM	y^x
EE	(	)	÷	GTO	×
SBR	−	RST	+	R/S	.
+/−	=	CLR	INV	log	CP
tan	Pgm	P→R	sin	cos	CMs
Exc	Prd	|x|	Eng	Fix	Int
Deg	Pause	x=t	Nop	Op	Rad
Lbl	x≥t	Σ+	$\bar{x}$	Grad	St flg
If flg	D.MS	π	List	Write	Dsz
Adv	Prt				

FLAGS / DRAPEAUX: 0 1 2 3 4 5 6 7 8 9

TITLE / TITRE Depth Equation Solution PAGE 2 OF / DE 3

TI PROGRAMMABLE

CODING FORM

FEUILLE DE PROGRAMMATION

155a

PROGRAMMER / PROGRAMMEUR D. Holmgren DATE Jan. 27'84

LOC ADR	CODE	KEY TOUCHE	COMMENTS COMMENTAIRES
000	71	SBR	
001	11	A	
002	43	RCL	
003	05	05	
004	42	STO	
005	04	04	
006	43	RCL	
007	03	03	
008	33	X²	
009	22	INV	
010	49	PRD	
011	04	04	
012	43	RCL	
013	00	00	
014	65	×	
015	02	2	
016	95	=	
017	49	PRD	
018	04	04	
019	01	1	
020	75	-	
021	43	RCL	
022	00	00	
023	95	=	
024	65	×	
025	03	3	
026	95	=	
027	44	SUM	
028	04	04	
029	03	3	
030	75	-	
031	43	RCL	
032	00	00	
033	95	=	
034	22	INV	
035	49	PRD	
036	04	04	
037	43	RCL	
038	01	01	
039	55	÷	
040	53	(	
041	43	RCL	
042	02	02	
043	65	×	
044	43	RCL	
045	04	04	
046	54	)	
047	95	=	
048	34	ΓX	
049	42	STO	
050	03	03	
051	91	R/S	
052	81	RST	
053	76	LBL	
054	11	A	
055	53	(	
056	53	(	
057	01	1	
058	75	-	
059	43	RCL	
060	03	03	
061	54	)	
062	65	×	
063	43	RCL	
064	03	03	
065	95	=	
066	34	ΓX	
067	94	+/-	
068	42	STO	
069	05	05	
070	03	3	
071	75	-	
[illegible]	53	(	
[illegible]	04	4	
[illegible]	65	×	
075	43	RCL	
076	03	03	
077	54	)	
[illegible]	95	=	
[illegible]	49	PRD	
080	05	05	
081	01	1	
[illegible]	55	÷	
[illegible]	53	(	
[illegible]	02	2	
085	65	×	
086	43	RCL	
087	03	03	
088	54	)	
089	95	=	
[illegible]	49	PRD	
[illegible]	05	05	
[illegible]	43	RCL	
[illegible]	03	03	
[illegible]	34	ΓX	
[illegible]	22	INV	
096	38	SIN	
097	65	×	
098	03	3	
099	95	=	
100	44	SUM	
101	05	05	
102	02	2	
103	55	÷	
104	53	(	
105	03	3	
106	65	×	
107	89	π	
[illegible]	54	)	
[illegible]	95	=	
110	49	PRD	
111	05	05	
112	54	)	
113	92	RTN	

MERGED CODES / TOUCHES COMBINEES

62 Pgm Ind	72 STO Ind	83 GTO Ind
63 Exc Ind	73 RCL Ind	84 Op Ind
64 Prd Ind	74 SUM Ind	92 INV SBR

TEXAS INSTRUMENTS

TITLE / TITRE: Depth Equation Solution PAGE 3 OF 3

TI PROGRAMMABLE
PROGRAM RECORD
FICHE PROGRAMME

PROGRAMMER / PROGRAMMEUR: D. Holmgren DATE: Jan.27'84

PARTITIONING (OP 17) / PARTITION (OP 17): 4 7 95 9
LIBRARY MODULE / MODULE ENFICHABLE: -
PRINTER / IMPRIMANTE: -
CARDS / CARTES: 1

PROGRAM DESCRIPTION • DESCRIPTION DU PROGRAMME

Example: HS Herculis

$x_b=0.6$, $\lambda_b=0.705$, so $1-\lambda_b=0.295$, $\lambda_a=0.9$ initial k=0.5

final k=0.549410 (4 iterations)

USER INSTRUCTION • MODE D'EMPLOI

STEP SEQUENCE	PROCEDURE	ENTER INTRODUIRE	PRESS APPUYER SUR			DISPLAY AFFICHAGE
0	Load Program					
1	Enter data	$x_b=0.6$	0.6	STO	00	0.6
		$1-\lambda_b=$ 0.295	0.295	STO	01	0.295
		$\lambda_a=0.9$	0.9	STO	02	0.9
		$k_{init}=0.5$	0.5	STO	03	0.5
2	Set to radian mode		2nd	RAD		
3	Begin iteration		r/s			
4	New k					0.5520....
5	Continue iteration		r/s			
	Next k value					0.54929....
	etc.					
	Fourth iteration					0.549410....
	(No change in subsequent iterations)					

USER DEFINED KEYS / TOUCHES UTILISATEUR	
A	$l_\tau(k)$ routine
B	
C	
D	
E	
A'	
B'	
C'	
D'	
E'	

DATA REGISTERS / REGISTRES-MEMOIRE (INV List)

0	x_b	0	
1	$1-\lambda_b$	1	
2	λ_a	2	
3	k	3	
4	Y(k,-1)	4	
5	$l_\tau(k)$	5	
6		6	
7		7	
8		8	
9		9	

LABELS (OP 08)

INV lnx CE CLR x⇌t x^2 √x 1/x STO RCL SUM y^x EE ( ) ÷ GTO × SBR − RST + R/S . +/− = CLR INV log CP tan Pgm P→R sin cos CMs Exc Prd |x| Eng Fix Int Deg Pause x=t Nop Op Rad Lbl x≥t Σ+ x̄ Grad St flg If flg D.MS π List Write Dsz Adv Prt

FLAGS / DRAPEAUX: 0 1 2 3 4 5 6 7 8 9

```
/COMPILE NOEXT NOLIST
C PROGRAM 'ZAPP.DC' FOR THE LEAST-SQUARES SOLUTION OF
C M LINEAR EQUATIONS IN N UNKNOWNS USING NORMAL EQUATIONS.
C THIS PROGRAM FOR DIFFERENTIAL CORRECTIONS USING IRWIN'S TABLES.
C BY D. HOLMGREN, OCT.14'83. SLIGHTLY MODIFIED - OCT. 20'83
C
      REAL T1,T2,WT,PV,SD,R1,R2,INCL,RK,AL,P,DARK,TAU,XTAU,PI
      INTEGER NTYPE
      DIMENSION A(81),B(3321),IP(81)
      DATA B/3321*0.0/,IP/81*1/
      PI=3.14159
C
C SPECIFY THE NUMBER OF EQUATIONS (M) AND THE NUMBER OF
C UNKNOWNS (N).
C
      READ, M,N,NTYPE
      READ,R1,R2,RK,INCL,AL,DARK
      INCL=INCL*PI/180.
C
      IF(M .LT. N) GO TO 16
      NP=N+1
C
C FORMATION OF NORMAL EQUATIONS MATRIX (B).
C
      DO 1 I=1,M
C
C READ IN NUMERICAL COEFFICIENTS, A, OF AN OBSERVATIONAL
C EQUATION:
C A(1)*X1+A(2)*X2+A(3)*X3+...+A(N)*XN=A(NP)
C AND ITS WEIGHT, WT.
C (IF NO WEIGHTING, SET WT=1.0 FOR ALL EQUATIONS).
C
      READ,(A(J),J=1,NP),WT,P
C TRANSIT ECLIPSE.
      IF(NTYPE-1)22,23,22
   22 TAU=2.*(ARSIN(SQRT(RK))+(4.*RK-3.)*(2.*RK+1)*SQRT(RK*(1.-RK))/3.
     -)/PI
      XTAU=3.*(1.-DARK)*RK*RK/(3.-DARK)+2.*DARK*TAU/(3.-DARK)
      A(1)=-A(1)*XTAU
      A(2)=-A(2)*AL*RK/R1
      A(3)=-A(3)*AL*RK/R2
      A(4)=A(4)*AL*RK*COS(P)*COS(P)/(R1*R2)
      GO TO 24
C OCCULTATION ECLIPSE.
   23 A(1)=-A(1)
      A(2)=-A(2)*AL/R1
      A(3)=-A(3)*AL/R2
      A(4)=A(4)*AL*COS(P)*COS(P)/(R1*R2)
   24 CONTINUE
C
      L=1
      DO 1 J=1,NP
      T1=A(J)*WT
      DO 1 K=J,NP
      B(L)=B(L)+T1*A(K)
    1 L=L+1
C
C INVERT NORMAL EQUATIONS.
C
      DO 12 I=1,N
```

```
      PV=0.0
      K=1
      DO 2 J=1,N
      IF(IP(J).EQ.-1) GO TO 2
      IF(PV.GE.B(K)) GO TO 2
      PV=B(K)
      I1=K
      L=J
    2 K=K+N-J+2
      IP(L)=-1
      PV=1.0/PV
      B(I1)=1.0
      I2=0
      DO 11 J=1,NP
      IF(J-L)3,11,4
    3 I3=I2+L
      T1=B(I3)*PV
      GO TO 5
    4 T1=B(I1+J-L)*FLOAT(IP(J))*PV
    5 I4=I2
      DO 10 K=J,NP
      IF(K-L)6,7,8
    6 T2=B(I4+L)*FLOAT(IP(K))
      I4=I4+NP-K
      GO TO 9
    7 B(I3)=0.0
    8 T2=B(I1+K-L)
    9 I5=I2+K
   10 B(I5)=B(I5)-T1*T2
   11 I2=I2+NP-J
      DO 12 K=L,NP
      B(I1)=B(I1)*PV
   12 I1=I1+1
C
      IF(M.EQ.N) GO TO 14
C
C EVALUATE STANDARD DEVIATION AND OUTPUT RESULTS.
C
C THE X'S ARE THE CORRECTIONS.
      T1=B(I2+NP)/FLOAT(M-N)
      J=1
      DO 13 I=1,N
      SD=SQRT(T1*B(J))
      K=J+NP-I
      WRITE(6,100)I,B(K),SD
  100 FORMAT('  X(',I3,')=',E15.7,5X,'S.D. =',E15.7)
   13 J=K+1
      WRITE(6,200) B(J)
  200 FORMAT(/' SUM OF SQUARES OF RESIDUALS =',E15.7)
      STOP
C
C OUTPUT RESULTS FOR M=N.
C
   14 K=NP
      DO 15 I=1,N
      WRITE(6,300)I,B(K)
  300 FORMAT('  X(',I3,')=',E15.7)
   15 K=K+NP-1
      STOP
C
```

```
C ERROR MESSAGE (M.LT.N)
C
  16 WRITE(6,400)M,N
 400 FORMAT('   EQUATIONS UNDERDETERMINED; NO SOLUTION.','
    -          ','   M=',I4,5X,'N=',I4)
     STOP
     END
/EXECUTE
15,4,1
0.210,0.122,0.582,84.63,0.100,0.60
1.000,0.000,0.000,0.000,0.0020,1.0,3.1460
0.9482,0.434,-0.365,0.399,0.0018,1.0,3.1724
0.9306,0.477,-0.384,0.419,0.0025,1.0,3.1862
0.9657,0.378,-0.331,0.367,0.0019,1.0,3.2126
0.8805,0.570,-0.409,0.448,0.0017,1.0,3.2340
0.4857,0.714,-0.108,0.352,0.0010,1.0,3.2591
0.5398,0.721,-0.165,0.374,0.0011,1.0,3.2673
0.4776,0.712,-0.0988,0.349,0.0010,1.0,3.2918
0.4051,0.688,-0.0213,0.318,0.0008,1.0,3.3018
0.5133,0.719,-0.137,0.364,0.0010,1.0,3.3326
0.3510,0.664,0.0361,0.294,0.0007,1.0,3.3640
0.2959,0.632,0.0923,0.268,0.0006,1.0,3.3778
0.1208,0.459,0.226,0.170,0.0002,1.0,3.4099
0.0650,0.359,0.232,0.126,0.0001,1.0,3.4281
0.0000,0.000,0.000,0.000,0.0000,1.0,3.4576
/END
END OF FILE
```

```
/COMPILE NOEXT NOLIST
1          /COMPILE NOEXT NOLIST
  X(  1)= -0.1982740E-02      S.D. =   0.1821888E-03
  X(  2)=  0.6007678E-03      S.D. =   0.1540373E-02
  X(  3)= -0.1127133E-04      S.D. =   0.8912557E-03
  X(  4)=  0.1697175E-03      S.D. =   0.4322550E-03
 SUM OF SQUARES OF RESIDUALS =   0.3706959E-06
END OF FILE
```

```
NPTS= 15 NTERMS=  4 MODE=  0

PARAMETERS AND INCREMENTS:
      0.2100000      0.0100000
      0.1220000      0.0100000
      1.4771000      0.1000000
      0.9000000      0.0010000
      1.0000000      0.0
      0.6000000      0.0
      0.6000000      0.0
ITERATION:        CHI-SQUARE:

  1           0.60393691E+00
NEW PARAMETERS AND FITTED VALUES;

      0.2111225      0.3542341
      0.1231265      0.3482378
      1.4817168      0.2529887
      0.4571903      5.5833859
      0.8445995      0.0574005
      0.8455732      0.0614268
      0.8474686      0.0612314
      0.8548371      0.0504629
      0.8642265      0.0494735
      0.8781773      0.0742227
      0.8832317      0.0638683
      0.8992648      0.0539352
      0.9060666      0.0542334
      0.9272828      0.0224172
      0.9483727      0.0172273
      0.9571829      0.0138171
      0.9759211      0.0121789
      0.9850607      0.0085393
      0.9964976      0.0035024
END OF FILE
```

```
15,4,0,1,1,
0.210,0.010
0.122,0.010
1.4771,0.100
0.9000,0.001
1.000,0.000
0.6000,0.000
0.6000,0.000
0.0000,0.9020,0.001
0.0264,0.9070,0.001
0.0402,0.9087,0.001
0.0666,0.9053,0.001
0.0880,0.9137,0.001
0.1131,0.9524,0.001
0.1213,0.9471,0.001
0.1458,0.9532,0.001
0.1558,0.9603,0.001
0.1866,0.9497,0.001
0.2180,0.9656,0.001
0.2318,0.9710,0.001
0.2639,0.9881,0.001
0.2821,0.9936,0.001
0.3116,1.0000,0.001
END OF FILE
```

Appendix 2 Program for Kitamura's Method

The program LCFT2 uses the procedure outlined in the chapter on Kitamura's method to compute the values of E, F_1, F_2, and F_1/F_2 (the characteristic functions). LCFT2 is a WATFIV program (a listing and sample run follows), and is therefore initiated by

$R *WATFIV SCARDS = LCFT2 .

As with ZAPP.DC, all results are printed out at the terminal. If one wanted to save the results in a file, one would follow the SCARDS command with SPRINT = an output file. The only input data required is the set of observations (half of one eclipse). The integer variable INORM, which is an input parameter, allows one to either normalize the light curve using equation (3.5) (INORM = 1) or to omit this step (INORM $\neq$ 1). The author's experience is that there is little difference in the values of E, F_1, F_2, and F_1/F_2 when INORM is used. One must also enter the last two values of the phase θ if INORM is to be used (in fact, these two values of θ must be included regardless of whether or not INORM is used). Hence, the first input line reads:

number of observations, θ_1, θ_2, INORM .

The observations in (θ, ℓ) pairs follow immediately. LCFT2 prints out all of the observations, followed by the values of the transforms (S_n and C_n), and finally the values of E, F_1, F_2, and F_1/F_2. All computations are done using double precision (REAL*8) arithmetic.

```
/COMPILE NOEXT NOLIST
C PROGRAM TO COMPUTE FOURIER TRANSFORM OF  LIGHT CURVE.
C THE TRANSFORMS COMPUTED ARE KITAMURA'S INCOMPLETE TRANSFORMS.
C DATA IS ENTERED IN PHASE-LUMINOSITY PAIRS AFTER /EXECUTE.
C BEFORE THE DATA APPEARS, THE NUMBER OF OBSERVATIONS MUST APPEAR.
C NOTE THAT THIS NUMBER IS AN INTEGER.
C PROGRAM IS INITIATED BY: $R *WATFIV SCARDS=LCFT2 SPRINT=<OUTPUT FILE>.
C TO SEE THE RESULTS, LIST THE OUTPUT FILE.
      REAL*8 S(3),C(3),PHASE(20),L(20),SSUM,CSUM,ST,CT,DP,
     -EPS,PHI,LAM,LS1,LS2,LC1,LC2,EE,F,F1,F2,LC0,AV,AK,C01,C02
      INTEGER N,K,I,NOBS,J,INORM
      CHARACTER*15 NAME
      N=0
C ENTER AND ECHO INPUT, WITH A HEADER.
      READ,NAME
      PRINT,NAME
      READ,NOBS,INORM,PHI1,PHI2
      READ,(PHASE(I),L(I), I=1,NOBS)
      PRINT,'OBSERVATIONS'
      PRINT,(PHASE(I),L(I), I=1,NOBS)
      PRINT,'FOURIER TRANSFORM OF LIGHT CURVE'
      PRINT,'NUMBER OF OBSERVATIONS',NOBS
      AV=0.0
      K=1
      WHILE(K .LE. NOBS)DO
      AV=AV+PHASE(K)
      K=K+1
      ENDWHILE
      AV=AV/NOBS
      EPS=PHASE(1)
      PHI=PHASE(NOBS)
      LAM=L(1)
      PRINT,'EPS=',EPS,'PHI=',PHI,'LAMBDA=',LAM
C COMPUTE TRANSFORM.
      IF(INORM.EQ.1)AK=1.
      IF(INORM.EQ.1)GO TO 5
      C01=0.
      DO 1 I=1,NOBS
      IF(PHASE(I).GT.PHI1)GO TO 2
    1 C01=C01+L(I)
    2 CONTINUE
      C02=0.
      DO 3 J=1,NOBS
      IF(PHASE(J).GT.PHI2)GO TO 4
    3 C02=C02+L(J)
    4 CONTINUE
      AK=(PHI1-PHI2)/(C01-C02)
    5 CONTINUE
      WHILE(N .LE. 2)DO
      I=1
      SSUM=CSUM=0.0
      K=NOBS-1
      WHILE(I .LE. K)DO
      DP=PHASE(I+1)-PHASE(I)
      ST=(L(I)*(DSIN(N*PHASE(I)))+L (I+1)*(DSIN(N*PHASE(I+1))))*DP
      CT=(L(I)*(DCOS(N*PHASE(I)))+L (I+1)*(DCOS(N*PHASE(I+1))))*DP
      SSUM=SSUM+ST
      CSUM=CSUM+CT
      I=I+1
      ENDWHILE
```

```
              S(N+1)=SSUM/2.0
              C(N+1)=CSUM/2.0
      C PRINT RESULTS.
              PRINT,'FOR N=',N,'SN=',S(N+1),'CN=',C(N+1)
              N=N+1
              ENDWHILE
              S(2)=S(2)+((L(1)-L(NOBS)*DCOS(PHI))*AV*AV/12.0)
              S(3)=S(3)+((L(1)-L(NOBS)*DCOS(2.0*PHI))*AV*AV/6.0)
              C(2)=C(2)+(AV*AV*L(NOBS)*DSIN(PHI)/12.0)
              C(3)=C(3)+(AV*AV*L(NOBS)*DSIN(PHI)/6.0)
              PRINT,'CORRECTED S2,S3,C2,C3 ARE:'
              PRINT,S(2),S(3),C(2),C(3)
              LS1=1.0-DCOS(PHI)-(1.0-DCOS(EPS))*LAM-S(2)*AK
              LS2=(1.0-DCOS(2.0*PHI)-(1.0-DCOS(2.0*EPS))*LAM)/2.0-S(3)*AK
              LC0=PHI-(EPS*LAM)-C(1)*AK
              LC1=DSIN(PHI)-(DSIN(EPS))*LAM-C(2)*AK
              LC2=(DSIN(2.0*PHI)-(DSIN(2.0*EPS))*LAM)/2.0-C(3)*AK
      C COMPUTE KITAMURA'S CHARACTERISTIC QUANTITIES.
              F1=LS1/LC1
              F2=LS2/LC2
              EE=LC0/(1.-LAM)
              F=F1/F2
              PRINT,'E=',EE
              PRINT,'F1=',F1
              PRINT,'F2=',F2
              PRINT,'F1/F2=',F
              STOP
              END
      /EXECUTE
      'HS HERCULIS'
      17,1,0.4932,1.0030
      0.0195,0.6500
      0.0352,0.6603
      0.0754,0.6639
      0.0886,0.6719
      0.1659,0.7567
      0.1791,0.7708
      0.2036,0.8093
      0.2168,0.8221
      0.2300,0.8405
      0.2457,0.8593
      0.2803,0.9031
      0.3443,0.9677
      0.3575,0.9703
      0.3846,0.9856
      0.4750,0.9911
      0.4882,0.9957
      0.5498,1.0030
      /END
) OF FILE
```

```
/COMPILE NOEXT NOLIST
1          /COMPILE NOEXT NOLIST
W DELPHINI
OBSERVATIONS
      0.7799999999999998D-02      0.7829999999999999D-01 0.1620D-01   0.83599D-01
      0.3410000000000000D-01      0.8829999999999999D-01 0.4650D-01   0.89099D-01
      0.6010000000000000D-01      0.1023000000000000D 00 0.7320D-01   0.11320D 00
      0.8619999999999999D-01      0.1337000000000000D 00 0.9840D-01   0.16830D 00
      0.1123000000000000D 00      0.2004000000000000D 00 0.1225D 00   0.23440D 00
      0.1354000000000000D 00      0.2547000000000000D 00 0.1500D 00   0.29240D 00
      0.1629000000000000D 00      0.3420000000000000D 00 0.1766D 00   0.39990D 00
      0.1889000000000000D 00      0.4305000000000000D 00 0.2021D 00   0.50820D 00
      0.2145000000000000D 00      0.5224000000000000D 00 0.2280D 00   0.58880D 00
      0.2415000000000000D 00      0.6281000000000001D 00 0.2[illegible]38D 00   0.69500D 00
      0.2680000000000000D 00      0.7482000000000000D 00 0.2820D 00   0.74820D 00
      0.2931000000000000D 00      0.8054000000000000D 00 0.3066D 00   0.85110D 00
      0.3277000000000000D 00      0.9078000000000000D 00 0.3540D 00   0.92470D 00
      0.3675000000000000D 00      0.9583000000000000D 00
FOURIER TRANSFORM OF LIGHT CURVE
NUMBER OF OBSERVATIONS          27
EPS=       0.7799999999999998D-02 PHI=       0.3675000000D 00 LAMBDA= 0.7829D-01
FOR N=           0 SN=     0.0000000000000000D 00 CN= 0.1680715599D 00
FOR N=           1 SN=     0.4218441752161736D-01 CN= 0.1620924353D 00
FOR N=           2 SN=     0.8087057448177471D-01 CN= 0.1446985135D 00
CORRECTED S2,S3,C2,C3 ARE:
      0.3999987750836892D-01      0.7748256693065065D-01 0.1630D 00   0.1465D 00
E=      0.2157076055115548D 00
F1=      0.1368155924971311D 00
F2=      0.2742454878427764D 00
F1/F2=      0.4988800128429707D 00
END OF FILE
```

Appendix 3 Programs for Kopal's Method

There are six programs for Kopal's method: TOTAL (total eclipses), ANNULAR (annular eclipses), PARTIAL (partial eclipses), EB.FS (least-squares Fourier analysis and moment determination), KAL (Kalman filter for moment determination), and ERROR (error analysis for orbital elements). Programs EB.FS and KAL have been adapted from the programs published by Jurkevich (1981). All six programs use the FORTRAN language, and are used in the following way:

```
SET T = 1.0
$R *FORTG SCARDS = PROGRAM NAME  T = 1.0
(computer responds with a message)
$R - LOAD#  5 = INPUT DATA FILE  6 = -P
(another message).
```

To see the results, list the temporary file -P. Listings of all programs, as well as sample input and output files follow. The programs TOTAL, ANNULAR, and PARTIAL all require the moments of the light curve as input data. In all cases, a value of the limb darkening is required. ANNULAR and PARTIAL require the depths of both minima as input data. ANNULAR uses the iterative method devised by Jurkevich (1970, pg. 75) for determining k (see chapter on H S Herculis for a complete description. PARTIAL requires an initial estimate of k, which is best determined by forming (according to Kopal (1979, pg. 176))

$$g_2 = \frac{A_2^2}{A_o A_4} \quad \text{and} \quad g_4 = \frac{A_4^2}{A_2 A_6} ,$$

where $g_2 = g_2(a,c_o)$ and $g_4 = g_4(a,c_o)$. The parameters a and c_o are defined as (r_1 = radius of eclipsed star, r_2 = radius of eclipsing star):

$$a = \frac{r_1}{r_1 + r_2} \qquad c_o = \frac{\cos i}{r_1 + r_2} .$$

One may determine a and c_o using the set of tables published by Demircan (1981, pp. 144-147). Another parameter $b = 1-a$ may be used to find k:

$$k = \frac{r_1}{r_2} = \frac{a}{b} .$$

This will be either greater than or less than one according to the eclipse type. To establish the eclipse type in the case of a partial eclipse, one may take advantage of a useful property of the Russell-Merrill χ-functions

$$\chi^{oc}(n=0.8) > \chi^{tr}(n=0.8)$$

where

$$n = \frac{1-\ell}{1-\ell_o} \quad (\ell_o = \text{value of } \ell \text{ at minimum}).$$

The χ-functions are easily computed with the equation

$$\chi = \frac{\sin^2\theta(n)}{\sin^2\theta(n=0.5)} .$$

The χ-functions allow for a quick determination of the eclipse type. Table A3.1 summarizes the input data required (in the correct format) by TOTAL, ANNULAR, and PARTIAL.

Program EB.FS requires only the ascending branch of one eclipse as input data, along with some out-of-eclipse points for the determination of Δu. EB.FS requires the angle of external tangency, as well as a number of integer numbers for controlling the process of calculation. The format shown in the sample input file is the one which will be most commonly used. If the moments are not to be computed, IYN in line 71 should be set equal to 1. The only other parameters of interest are MI, the number of in-eclipse observations, MMAX, the number of out-of-eclipse observations, and NMAX, the number of Fourier coefficients to be used. These three numbers appear on line 1 in the following order:

NMAX, MI, MMAX .

The value of θ at external tangency follows these numbers. Lines 3 need not be altered in most cases. The last two lines of the EB.FS input file read:

θ(external tangency), 1, 7, 1, 7,

1.0E-06,

where 1 and 7 are digits controlling the number of moments to be calculated, and 1.0E-06 is the maximum allowable error in the calculation of the moments. Program KAL is quite simple to use. The first line of the input file contains the value of q/r, which is denoted by T in the program (q/r = intensity of white noise/variance of variable), the value of $1-\ell$ at zero phase (estimated), and the number of observations. In other words:

T,1-ℓ(0), number of observations.

The data follow in (θ,ℓ) pairs, where θ must be in radians. Program error requires a special command for execution. Since it incorporates in IMSL subroutine, the following command should be used:

$R - LOAD# +*IMSLLIB 5 = input file, 6 = -P.

The input format required is shown in table A3.1.

Table A3.1 Input Formats for Programs Using Kopal Method.

TOTAL:	A_o, A_2, A_4, A_6	(format 4E15.6)
	x, L_1	(format 2F14.7)
ANNULAR:	x, DOC, DTR, L_1	(format 4F14.7)
	A_o, A_2, A_4, A_6	(format 4E15.6)

(Note DOC = value of ℓ at minimum of occultation,
DTR = value of ℓ at minimum of transit).

PARTIAL:	x, DOC, DTR, k(estimated)	(format 4F14.7)
	A_o, A_2, A_4, A_6	(format 4E15.6)
ERROR:	ΔA_o^{tr}, ΔA_o^{oc}, ΔA_2, ΔA_4, ΔA_6, NTYPE	(format 5E15.6, I3)
	r_1, r_2, X, Y, i, k, L_1, x	(format 8F14.7)
	A_o^{oc}, A_o^{tr}, α_o	(format 3F14.7)

where: tr = transit

oc = occultation

$$\text{NTYPE} = \begin{cases} 1 & \text{occultation} \\ \text{anything else} & \text{transit} \end{cases}$$

r_1 = eclipsed star

r_2 = eclipsing star

$X = r_2^2 \csc^2 i$

$Y = \cot^2 i$

α_o = value of α at minimum of primary eclipse

```
C PROGRAM TO ESTIMATE MOMENTS OF BINARY LIGHT CURVES FROM
C FOURIER COSINE SERIES APPROXIMATION OF LIGHT LOSS.
C THIS PROGRAM ADAPTED (FROM ARTICLE BY JURKEVICH) BY
C D. HOLMGREN FOR USE ON MTS FORTRAN. NOV. 19'83.
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION T(400),EL(400),RU(13,13),RL(13,13),Q(13,13),
     -AER(13),CS(13),SN(13),X(13),APHI(9,9),AMOM(13),AMOMER(13),
     -EPS(13)
C DIMENSION OF 13 ON RU AND RL IMPLIES 13 TERMS IN THE REGRESSION
C EQUATION:10 HARMONICS,A0,(-DELU),AND THE ABSOLUTE TERM.
C
      READ(5,13) NMAX,MI,MMAX
   13 FORMAT(I2,2I3)
      READ(5,14) THETAT
   14 FORMAT(F14.7)
      READ(5,499) I2,I3
  499 FORMAT(2I2)
C THETAT IS THE POINT OF EXTERNAL CONTACT IN RADIANS.
C IF I2=1 DO NOT COMPUTE INDIVIDUAL RESIDUALS FIRST TIME AROUND.
C IF I3=1 (AUTOMATICALLY SETS I1=0) COMPUTE RESIDUALS.
C NMAX=HIGHEST HARMONIC USED, MI=NUMBER OF OBSERVATIONS WITHIN
C ECLIPSE, MMAX=NUMBER OF OBSERVATIONS OUTSIDE ECLIPSE.
C
      WRITE(6,800)
  800 FORMAT(1H1)
      WRITE(6,120) NMAX,MI,MMAX
  120 FORMAT(1X,5HNMAX=I2,2X,3HMI=,I3,2X,5HMMAX=I3,2X)
      WRITE(6,121)I2,I3
  121 FORMAT(1X,5H  I2=I2,2X,3HI3=I2)
      WRITE(6,122)THETAT
  122 FORMAT(1X,7HTHETAT=F8.6//)
      NOBS=MI+MMAX
      DO 15 I=1,NOBS
      READ(5,16) T(I),EL(I)
   16 FORMAT(2E15.6)
   15 CONTINUE
C
C DEFINE CONSTANTS.
      PI=3.14159265358979300
C
      IMAX=NMAX+2
      JMAX=NMAX+3
      REFANG=PI/THETAT
C
C COMPUTE LOSS OF LIGHT.
      DO 53 I=1,MI
      EL(I)=1.-EL(I)
   53 CONTINUE
C
C COMPUTE SUM OF SQUARES OF LIGHT LOSS
      DO 52 I=1,MI
      TRY1=TRY1+EL(I)*EL(I)
   52 CONTINUE
C
C ZERO OUT RU MATRIX.
      DO 1 J=1,IMAX
      DO 2 I=1,J
    2 RU(J,I)=0.
      RU(JMAX,J)=0.
```

```
    1 CONTINUE
C
C COMPUTATION OF COSINES FOR K'TH OBSERVATION.
   11 DO 10 M=1,MI
      ANG=REFANG*T(M)
      SSN=DSIN(ANG)
      CSN=DCOS(ANG)
      SN(1)=SSN
      CS(1)=CSN
      DO 200 J=2,NMAX
      SN(J)=SN(J-1)*CSN+CS(J-1)*SSN
  200 CS(J)=CS(J-1)*CSN-SN(J-1)*SSN
C
C COMPUTATION OF INDIVIDUAL RESIDUALS IF NEEDED.
      IF(I2-1)206,207,102
  206 Y=0.
      DO 501 I=2,NMAX
  501 Y=Y+Q(I,JMAX)*CS(I-1)
      Y=Y+0.5*Q(1,JMAX)-Q(IMAX,JMAX)
      S1=EL(M)-Y
      S3=S3+S1*S1
      WRITE(6,128)T(M),EL(M),Y,S1
  128 FORMAT(10X,D23.16,3(3XD23.16))
      I3=0
      GO TO 10
C
C COMPUTATION OF COEFFICIENTS OF NORMAL EQUATIONS.
  207 X(1)=0.5
      NU=NMAX+1
      DO 201 J=2,NU
      X(J)=CS(J-1)
  201 CONTINUE
      X(JMAX)=-EL(M)
      DO 203 J=1,NU
      DO 202 I=1,J
  202 RU(J,I)=RU(J,I)+X(I)*X(J)
      RU(JMAX,J)=RU(JMAX,J)+X(J)*X(JMAX)
  203 CONTINUE
   10 CONTINUE
      IF(I2.EQ.1)GO TO 41
      WRITE(6,127)S3
  127 FORMAT(7X,3HS3=D23.16)
   41 DO 500 J=1,IMAX
      RU(IMAX,J)=-2.*RU(J,1)
  500 CONTINUE
      RU(IMAX,IMAX)=RU(IMAX,IMAX)+MMAX
      RU(JMAX,IMAX)=-2.*RU(JMAX,1)
C
C SOLUTION OF  NORMAL EQUATIONS.
      NP1=JMAX
      CALL CRACOV(RU,NP1,RL,Q)
      WRITE(6,124)
  124 FORMAT(//)
      WRITE(6,125)
  125 FORMAT(2X,1HJ,3X,1HI,12X,2HRU,24X,2HRL,18X,1HI,3X,1HJ,
     -12X,1HQ/)
      DO 505 I=1,JMAX
      DO 504 J=I,JMAX
      WRITE(6,123)J,I,RU(J,I),RL(J,I),I,J,Q(I,J)
  123 FORMAT(1X,I2,2X,I2,3X,D23.16,3X,D23.16,5X,I2,2X,I2,3X,D23.16)
```

```
  504 CONTINUE
      WRITE(6,506)
  506 FORMAT(/)
  505 CONTINUE
C COMPUTE THE SECOND TERM OF EQUATION (15).
      TRY2=0.
      DO 36 I=1,IMAX
      TRY2=TRY2+RL(JMAX,I)*RL(JMAX,I)
   36 CONTINUE
      DEN=MI+1-IMAX
      S2=(TRY1-TRY2)/DEN
      ERRMEA=DSQRT(S2)
C
C COMPUTE ERRORS OF UNKNOWNS.
      DO 33 I=1,IMAX
      AER(I)=0.
   33 CONTINUE
      DO 35 I=1,IMAX
      DO 32 J=I,IMAX
   32 AER(I)=AER(I)+Q(I,J)*Q(I,J)
      AER(I)=ERRMEA*DSQRT(AER(I))
   35 CONTINUE
C
C PRINT UNKNOWNS AND THEIR ERRORS
      WRITE(6,124)
      WRITE(6,38)
   38 FORMAT(6X,1HI,12X,4HA(I),22X,6HAER(I)/)
      DO 37 I=1,IMAX
      I1=I-1
      WRITE(6,39)I1,Q(I,JMAX),AER(I)
   39 FORMAT(5X,I2,2(3X,D23.16))
   37 CONTINUE
      WRITE(6,126)S2
  126 FORMAT(/7X,3HS2=D23.16//)
      DO 618 L=1,2
      WRITE(6,800)
  618 CONTINUE
C
C DECISION WHETHER TO COMPUTE INDIVIDUAL RESIDUALS.
      IF(I3-1)102,208,102
  208 I2=0
      S3=0.
      WRITE(6,110)
  110 FORMAT(20X,4HT(M),20X,5HEL(M),22X,1HY,25X,2HS1)
      GO TO 11
C
C COMPUTE MOMENTS IF IYN=0.
  102 READ(5,17) IYN
   17 FORMAT(I1)
      IF(IYN.EQ.1) GO TO 612
      READ(5,18) THETA,MUIN,MUFIN,IIN,IFIN
   18 FORMAT(F14.7,4I2)
      READ(5,19) EPS(1)
   19 FORMAT(E15.6)
      DO 617 K=2,9
      EPS(K)=EPS(K-1)*0.1D0
  617 CONTINUE
C
C NOTE THAT MU AND I ARE OFFSET. TO RUN MU=0 CASE, MUTOP MUST BE 1.
C THETA IS IN RADIANS.
```

```
      C
            DO 12 J=MUIN,MUFIN
            TOL=EPS(J)
            DO 12 I=IIN,IFIN
      C BELOW J1 IS MU,I1 IS THE RUNNING INDEX  IN PHI(I,MU).
            J1=J-1
            I1=I-1
            WRITE(6,103)J1,I1,EPS(J)
        103 FORMAT(1X,3HMU=I3,3X,2HI=I3,3X,4HEPS=D23.16/)
            CALL PHI(I1,J1,THETA,TOL,JJ,PHI1)
            WRITE(6,104)J1,I1,JJ,PHI1
        104 FORMAT(1X,3HMU=,I3,3X,2HI=I3,3X,3HJJ=,I6,3X,10HPHI(I,MU)=D23.16//)
            APHI(I,J)=PHI1
         12 CONTINUE
      C
      C PRINT TABLE OF PHI'S.
            WRITE(6,800)
            WRITE(6,614)
        614 FORMAT(2X,2HMU,2X,1HI,9X,10HAPHI(I,MU),11X,7HEPS(MU)/)
            DO 616 J=1,MUFIN
            DO 619 I=1,IFIN
            J1=J-1
            I1=I-1
            WRITE(6,615)J1,I1,APHI(I,J),EPS(J)
        615 FORMAT(2X,I2,2X,I1,3X,D23.16,3X,D8.2)
        619 CONTINUE
            WRITE(6,506)
        616 CONTINUE
      C COMPUTE MOMENTS FROM ASUBMU=SUM(APHI*NEUMANNNO*ASUBI)
            IFIN=IMAX-1
            DO 300 J=1,MUFIN
            I=1
            SUM=APHI(I,J)*Q(I,JMAX)
            DO 400 I=2,IFIN
            SUM=SUM+2.*APHI(I,J)*Q(I,JMAX)
        400 CONTINUE
            AMOM(J)=SUM
        300 CONTINUE
      C
      C COMPUTE ERRORS IN AMOM(I)
            DO 600 J=1,MUFIN
            SUM1=0.
            DO 602 I=1,IFIN
            SUM=0.
            DO 601 K=1,I
        601 SUM=SUM+APHI(K,J)*Q(K,I)
            SUM1=SUM1+SUM*SUM
        602 CONTINUE
            AMOMER(J)=ERRMEA*DSQRT(SUM1)
        600 CONTINUE
      C
      C PRINT A TABLE OF MOMENTS AND THEIR ERRORS.
            WRITE(6,800)
            WRITE(6,610)
        610 FORMAT(10X,2HMU,12X,6HMOMENT,22X,5HERROR/)
            DO 612 J=1,MUFIN
            MU=J-1
            WRITE(6,611)MU,AMOM(J),AMOMER(J)
        611 FORMAT(10X,I2,4X,D23.16,4X,D23.16)
        612 CONTINUE
```

```
      STOP
      END
      SUBROUTINE CRACOV(RU,NP1,RL,Q)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION RU(13,13),RL(13,13),Q(13,13)
      INTEGER DI
      N=NP1-1
      RL(NP1,NP1)=1.
      RL(1,1)=DSQRT(RU(1,1))
      DO 50 K=1,N
      KM1=K-1
      DO 40 I=K,NP1
      SUM=0.
      IF(KM1.NE.0)GO TO 10
      RL(I,K)=RU(I,K)/RL(1,1)
      GO TO 40
   10 DO 30 J=1,KM1
      SUM=SUM+RL(I,J)*RL(K,J)
   30 CONTINUE
      IF(K.EQ.I)GO TO 60
      RL(I,K)=(RU(I,K)-SUM)/RL(K,K)
      GO TO 70
   60 RL(I,K)=DSQRT(RU(I,K)-SUM)
   70 CONTINUE
   40 CONTINUE
   50 CONTINUE
      DO 100 I=1,N
      Q(I,I)=1./RL(I,I)
  100 CONTINUE
      Q(NP1,NP1)=1.
      DO 130 I=1,N
      K2=NP1-I
      DO 120 K=1,K2
      SUM=0.
      DO 110 J=1,K
      DI=J-1
      SUM=SUM+RL(I+K,I+DI)*Q(I,I+DI)
  110 CONTINUE
      Q(I,I+K)=-SUM/RL(I+K,I+K)
  120 CONTINUE
  130 CONTINUE
      RETURN
      END
      SUBROUTINE PHI(I,MU,THETA,TOL,JJ,PHI1)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL JY
      PI=3.1415926535900
      E=2.7182818284500
      ONE=FLOAT(-1)
      AMU=FLOAT(MU)
      FACT=(E**(1+MU))/PI
      CALL GAMMA(1.+AMU,GAM,IER)
      TS1=GAM/(2.D0**MU)
      AI=FLOAT(I)
      AIPI=AI*PI
      TS3=(AMU+3.D0)/2.D0
      TS4=(AMU+1.D0)/2.D0
      SUM=0.D0
      J=-1
C
```

```
C IS MU ODD OR EVEN
      IODD=1
      Y=AMU
      Y=Y/2.D0
      JY=SNGL(Y)
      IY=IFIX(JY)
      DIFF=Y-FLOAT(IY)
      IF(DABS(DIFF).LE.0.1D-10)IODD=0
C
C GENERAL FORM OF SERIES FOLLOWS.
C
   20 J=J+1
      ITS=MU+1-2*J
      IF(ITS.EQ.0)GO TO 10
      AJ=FLCAT(J)
      TS2=2.*AJ+1.
      ARG=TS2*THETA
      ARGSQ=ARG*ARG
      TEMPNU=(ONE**(I+J))*ARGSQ*DSIN(ARG)/(ARGSQ-AIPI*AIPI)
      IF((MU.GE.0).AND.(J.GT.50))GO TO 21
      CALL GAMMA(TS3+AJ,GAM1,IER)
      CALL GAMMA(TS4-AJ,GAM2,IER)
      FACT1=1./(GAM1*GAM2)
      GO TO 30
C
C ASYMPTOTIC SECTION FOR J GREATER THAN 50.
C
   21 A1=(AMU+1.)/(2.*AJ)
      A2=(AMU-1.)/(2.*AJ)
      ANUM=(1.-A1)**(J-MU/2)
      DENA=(1.-A2)**(J+1+MU/2)
      DENB=AJ**(1+MU)
      TEMP1=ANUM/(DENA*DENB)
      AMPL=FACT*TEMP1
      ANG=(AJ-AMU/2.+0.5D0)*PI
      FACT1=AMPL*DSIN(ANG)
C
C COMMON SECTION.
C
   30 TEMP=TEMPNU*FACT1
      SUM=SUM+TEMP
      IF(IODD.EQ.1)GO TO 20
      IF(DABS(TEMP).LE.TOL)GO TO 10
      GO TO 20
   10 CONTINUE
      JJ=J
      PHI1=TS1*SUM
      RETURN
      END
      SUBROUTINE GAMMA(XX,GX,IER)
C SUBROUTINE TO COMPUTE VALUES OF GAMMA FUNCTION.
C PARAMETERS:XX - THE ARGUMENT OF THE GAMMA FUNCTION.
C            GX - THE RESULTANT GAMMA FUNCTION VALUES.
C            IER - TOLERANCE CODE:
C                IER=0 NO TOLERANCE
C                   =1 XX IS WITHIN .000001 OF BEING
C                      A NEGATIVE INTEGER.
C                   =2 XX.GT.34.5,GX IS SET TO 1.E38.
      IMPLICIT REAL*8 (A-H,O-Z)
      IF(XX.LE.34.5D0) GO TO 6
```

```
      IER=2
      GX=1.0D+38
    6 X=XX
      TOL=1.0D-06
      IER=0
      GX=1.D0
      IF(X-2.D0)50,50,15
   10 IF(X.LE.2.D0) GO TO 110
   15 X=X-1.D0
      GX=GX*X
      GO TO 10
   50 IF(X-1.D0)60,120,110
C SEE IF X IS NEAR NEGATIVE INTEGER OR ZERO
   60 IF(X.GT.TOL)GO TO 80
      K=X
      Y=FLOAT(K)-X
      IF(DABS(Y).LE.TOL) GO TO 130
      IF((1.D0-Y).LE.TOL) GO TO 130
C X NOT NEAR NEGATIVE INTEGER OR ZERO.
   70 IF(X.GT.1.D0) GO TO 110
   80 GX=GX/X
      X=X+1.D0
      GO TO 70
  110 Y=X-1.D0
      GY=1.D0+Y*(-0.5771017D0+Y*(0.9858540D0+Y*(-0.8764218D0+
     -Y*(0.8328212D0+Y*(-0.5684729D0+Y*(0.2548205D0+Y*(-0.05149931D0)))
     -)))
      GX=GX*GY
  120 RETURN
  130 IER=1
      RETURN
      END
ND OF FILE
```

```
4,46,21,
0.4663,
1,0,
6.900E-03,6.630E-01
8.797E-03,6.558E-01
1.900E-02,6.520E-01
2.011E-02,6.650E-01
3.519E-02,6.630E-01
3.581E-02,6.668E-01
7.100E-02,6.730E-01
7.540E-02,6.662E-01
8.357E-02,6.748E-01
8.859E-02,6.742E-01
9.990E-02,6.842E-01
1.370E-01,7.010E-01
1.533E-01,7.191E-01
1.659E-01,7.593E-01
1.690E-01,7.610E-01
1.791E-01,7.730E-01
1.973E-01,8.128E-01
2.036E-01,8.121E-01
2.036E-01,8.091E-01
2.168E-01,8.249E-01
2.199E-01,8.241E-01
2.218E-01,8.279E-01
2.300E-01,8.433E-01
2.457E-01,8.620E-01
2.590E-01,8.760E-01
2.601E-01,8.760E-01
2.695E-01,8.980E-01
2.802E-01,9.061E-01
2.959E-01,9.256E-01
2.972E-01,9.179E-01
3.123E-01,9.428E-01
3.192E-01,9.341E-01
3.267E-01,9.367E-01
3.280E-01,9.629E-01
3.349E-01,9.603E-01
3.443E-01,9.710E-01
3.456E-01,9.701E-01
3.581E-01,9.736E-01
3.550E-01,9.745E-01
3.657E-01,9.745E-01
3.814E-01,9.881E-01
3.845E-01,9.890E-01
4.084E-01,9.917E-01
4.241E-01,9.917E-01
4.310E-01,9.982E-01
4.540E-01,9.917E-01
4.750E-01,9.945E-01
4.882E-01,9.991E-01
5.429E-01,9.927E-01
5.498E-01,1.006E 00
5.592E-01,9.963E-01
5.693E-01,9.936E-01
5.820E-01,9.908E-01
6.183E-01,9.963E-01
6.370E-01,9.890E-01
6.597E-01,9.972E-01
6.810E-01,9.927E-01
```

```
6.893E-01,1.001E 00
9.494E-01,9.927E-01
9.626E-01,9.972E-01
1.404E 00,9.954E-01
1.424E 00,1.011E 00
1.441E 00,1.006E 00
1.462E 0C,1.002E 00
1.497E 00,1.003E 00
1.510E 00,1.010E 00
1.529E 00,9.908E-01
0,
0.4663,1,7,1,7,
1.0E-06
END OF FILE
```

I

NMAX= 4 MI= 46 MMAX= 21

I2= 1 I3= 0

THETAT-0.466300

J	I	RU	RL
1	1	0.1150000000000000E+02	0.3391164991562634E+01
2	1	-0.2337417172564676E+00	[illegible]
3	1	-0.2295195577095679E+01	-0.6768162512901097E+00
4	1	0.3695987227737109E+01	[illegible]
5	1	0.4891359722293045E+00	[illegible]
6	1	-0.2300000000000000E+02	[illegible]
7	1	0.3690949999999996E+01	-0.1088401776139834E+01
2	2	0.2070480442290431E+02	0.4549731149945432E+01
3	2	[illegible]	0.7507245386857008E+00
4	2	[illegible]	-0.3804482621129172E+00
5	2	0.6417298947035033E+01	0.1412663870286626E+01
6	2	0.4574834345129352E+00	0.1220097394808122E-16
7	2	-0.3695872057906515E+01	-0.8288164390195610E+00
3	3	0.2348913597222927E+02	0.4739986118253303E+01
4	3	0.2487570002041458E+01	0.7406847006849913E+00
5	3	-0.2057209277416414E+01	-0.6434845563760302E+00
6	3	0.4590391154191357E+01	[illegible]
7	3	-0.6325235652425423E-01	[illegible]
4	4	[illegible]	0.4618092413823091E+01
5	4	[illegible]	-0.3608055608821379E+00
6	4	[illegible]	[illegible]
7	4	0.6862085380713169E+00	[illegible]
5	5	0.2448060243183533E+02	[illegible]
6	5	-0.9782719444585090E+00	-0.2667738247829130E-16
7	5	[illegible]	0.1584575464270696E-01
6	6	0.6700000000000000E+02	0.4582575694955840E+01
7	6	0.7381899599999991E+01	[illegible]
7	7	UUUUUUUUUUUUUUUUUUUUU	0.1000000000000000E+01

I	A(I)	AER(I)
0	0.3298649555337910E+00	[illegible]
1	0.1808710586346974E+00	[illegible]

I	J	Q
1	1	0.2948839123097943E+00
1	2	0.4457377532449905E-02
1	3	0.4139852929155914E-01
1	4	-0.7585549024577098E-01
1	5	-0.1058932026705827E-01
1	6	0.4364357804719847E+00
1	7	0.3298649555337910E+00
2	2	0.2197932069045428E+00
2	3	-0.3481110487312316E-01
2	4	0.2369030036234182E-01
2	5	-0.6927714769747591E-01
2	6	-0.1355067704999971E-18
2	7	0.1808710586346974E+00
3	3	0.2109710819930634E+00
3	4	-0.3383714285393836E-01
3	5	0.2638859062972673E-01
3	6	-0.2713057672470925E-17
3	7	0.9047302208186694E-02
4	4	0.2165395251072744E+00
4	5	0.1668749183178674E-01
4	6	0.4641114674013645E-17
4	7	-0.1022705708986199E-01
5	5	0.2135897774016940E+00
5	6	0.1243409070464465E-17
5	7	-0.3384491206897639E-02
6	6	0.2182178902359924E+00
6	7	-0.5286776307738839E-16
7	7	0.1000000000000000E+01

```
          2    0.9047302208186694E-02    0.1558257563163604E-02
          3   -0.1022705708986199E-01    0.1571943951827974E-02
          4   -0.3384491206897639E-02    0.1545946051006363E-02
          5   -0.5286776307738839E-16    0.1579443968588530E-02

           S2= 0.5238750824812443E-04



    1
    1
     MU=   0    I=   0    EPS= 0.1000000000000000E-05

     MU=   0    I=   0    JJ=    397    PHI(I,MU)= 0.4992063467509574E+00


     MU=   0    I=   1    EPS= 0.1000000000000000E-05

     MU=   0    I=   1    JJ=    397    PHI(I,MU)= 0.5007964662738852E+00


     MU=   0    I=   2    EPS= 0.1000000000000000E-05

     MU=   0    I=   2    JJ=    397    PHI(I,MU)= 0.4992061223261391E+00


     MU=   0    I=   3    EPS= 0.1000000000000000E-05

     MU=   0    I=   3    JJ=    397    PHI(I,MU)= 0.5007969150966195E+00


     MU=   1    I=   0    EPS= 0.9999999999999998E-07

     MU=   1    I=   0    JJ=      1    PHI(I,MU)= 0.2247921951182108E+00


     MU=   1    I=   1    EPS= 0.9999999999999998E-07

     MU=   1    I=   1    JJ=      1    PHI(I,MU)= 0.5063923716540568E-02


     MU=   1    I=   2    EPS= 0.9999999999999998E-07

     MU=   1    I=   2    JJ=      1    PHI(I,MU)=-0.1244947112091002E-02


     MU=   1    I=   3    EPS= 0.9999999999999998E-07

     MU=   1    I=   3    JJ=      1    PHI(I,MU)= 0.5516126340430748E-03


     MU=   2    I=   0    EPS= 0.9999999999999997E-08

     MU=   2    I=   0    JJ=    205    PHI(I,MU)= 0.1010716530804723E+00


     MU=   2    I=   1    EPS= 0.9999999999999997E-08

     MU=   2    I=   1    JJ=    205    PHI(I,MU)=-0.3856155094274944E-01
```

```
MU=  2   I=  2   EPS= 0.999999999999997E-08

MU=  2   I=  2   JJ=   205   PHI(I,MU)=-0.226820239263617 2E-02


MU=  2   I=  3   EPS= 0.999999999999997E-08

MU=  2   I=  3   JJ=   205   PHI(I,MU)=-0.395348346092003 6E-02


MU=  3   I=  0   EPS= 0.999999999999996E-09

MU=  3   I=  0   JJ=     2   PHI(I,MU)= 0.454363767781061 3E-01


MU=  3   I=  1   EPS= 0.999999999999996E-09

MU=  3   I=  1   JJ=     2   PHI(I,MU)=-0.266607585921012 0E-01


MU=  3   I=  2   EPS= 0.999999999999996E-09

MU=  3   I=  2   JJ=     2   PHI(I,MU)= 0.548954240204755 2E-02


MU=  3   I=  3   EPS= 0.999999999999996E-09

MU=  3   I=  3   JJ=     2   PHI(I,MU)=-0.236068203496229 3E-02


1
  MU  I           APHI(I,MU)            EPS(MU)

   0  0    0.499206346750957 4E+00   0.10E-05
   0  1    0.500796466273885 2E+00   0.10E-05
   0  2    0.499206122326139 1E+00   0.10E-05
   0  3    0.500796915096619 5E+00   0.10E-05


   1  0    0.224792195118210 8E+00   0.10E-06
   1  1    0.506392371654056 8E-02   0.10E-06
   1  2   -0.124494711209100 2E-02   0.10E-06
   1  3    0.551612634043074 8E-03   0.10E-06


   2  0    0.101071653080472 3E+00   0.10E-07
   2  1   -0.385615509427494 4E-01   0.10E-07
   2  2   -0.226820239263617 2E-02   0.10E-07
   2  3   -0.395348346092003 6E-02   0.10E-07


   3  0    0.454363767781061 3E-01   0.10E-08
   3  1   -0.266607585921012 0E-01   0.10E-08
   3  2    0.548954240204755 2E-02   0.10E-08
   3  3   -0.236068203496229 3E-02   0.10E-08


1
            MU               MOMENT                        ERROR
```

181	0	0.344619433427620 4E+00	0.162976054803083 0E-02
182	1	0.759490923606116 3E-01	0.499681842661356 0E-03
183	2	0.194304820429812 1E-01	0.236533947704418 6E-03
184	3	0.549116590248875 8E-02	0.114950563465715 0E-03

END OF FILE

```
C PROGRAM TO ESTIMATE MOMENTS OF BINARIES BY
C KALMAN FILTER - EVEN MOMENTS A(2M).
C THIS IS JURKEVICH'S PROGRAM.
C ADAPTED FOR USE ON MTS FORTRAN BY
C D. HOLMGREN, NOV.'83.
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION F(400),EL(400)
      INTEGER NOBS
      READ(5,1) T,X,NOBS
    1 FORMAT(2F14.7,I3)
      DO 2 I=1,NOBS
      READ(5,3) F(I),EL(I)
    3 FORMAT(2F14.7)
    2 CONTINUE
      P=1.
      P12=0.
      P14=0.
      P16=0.
      P18=0.
      X2=0.
      X4=0.
      X6=0.
      X8=0.
      B=0.
      DO 4 I=1,NOBS
      Z=1.-EL(I)
      DIFZX=Z-X
      A=B
      B=F(I)
      SA=DSIN(A)
      SB=DSIN(B)
      C=B-A
      AM=P+T*C
      SB2=SB*SB
      SA2=SA*SA
      SB4=SB2*SB2
      SA4=SA2*SA2
      SB6=SB2*SB4
      SB8=SB4*SB4
      DIF2=(SB+SA)*(SB-SA)
      SIG=SB2+SA2
      DIF4=SIG*DIF2
      SBSA=SB*SA
      DIF6=DIF2*(SIG-SBSA)*(SIG+SBSA)
      DIF8=(SB4+SA4)*SIG*DIF2
      AM12=P12+P*DIF2+T*(C*SB2+S2(A)-S2(B))
      AM14=P14+P*DIF4+T*(C*SB4+S4(A)-S4(B))
      AM16=P16+P*DIF6+T*(C*SB6+S6(A)-S6(B))
      AM18=P18+P*DIF8+T*(C*SB8+S8(A)-S8(B))
      DM=1./(1.+AM)
      P=AM*DM
      P12=AM12*DM
      P14=AM14*DM
      P16=AM16*DM
      P18=AM18*DM
      X2=X2+X*DIF2+P12*DIFZX
      X4=X4+X*DIF4+P14*DIFZX
      X6=X6+X*DIF6+P16*DIFZX
      X8=X8+X*DIF8+P18*DIFZX
    4 X=X+P*DIFZX
```

```
      WRITE(6,300)X2,X4,X6,X8
  300 FORMAT(1X,3HA2=E11.5,2X3HA4=E11.5,2X3HA6=E11.5,2X3HA8=E11.5//)
      STOP
      END
      REAL FUNCTION S2(X)
      REAL S2,X
      S2=(X-SIN(X)*COS(X))/2.
      RETURN
      END
      REAL FUNCTION S4(X)
      REAL  S4,X,SX,CX
      SX=SIN(X)
      CX=COS(X)
      S4=3.*X/8.-(3.*SX/8.+(SX*SX*SX)/4.)*CX
      RETURN
      END
      REAL FUNCTION S6(X)
      REAL S6,X,SX,CX,S3,S5
      SX=SIN(X)
      CX=COS(X)
      S3=SX*SX*SX
      S5=S3*SX*SX
      S6=5.*X/16.-(5.*SX/16.+5.*S3/24.+S5/6.)*CX
      RETURN
      END
      REAL FUNCTION S8(X)
      REAL S8,X,SX,CX,S3,S5,S7
      SX=SIN(X)
      CX=COS(X)
      S3=SX*SX*SX
      S5=S3*SX*SX
      S7=S5*SX*SX
      S8=35.*X/128.-(35.*SX/128.+35.*S3/192.+7.*S5/48.+S7/8.)*CX
      RETURN
      END
END OF FILE
```

```
C PROGRAM TO USE KOPAL'S FREQUENCY DOMAIN METHOD
C TO COMPUTE GEOMETRIC ELEMENTS OF AN ECLIPSING BINARY
C THIS VERSION FOR TOTAL ECLIPSES - PROGRAM 'TOTAL'.
C D. HOLMGREN, AUG. '83.
C REVISED MARCH'84 FOR USE WITH 'EB.FS'.
C
      IMPLICIT REAL(A-H,O-Z)
      DIMENSION C(3),CB(3)
C READ IN DATA.
      READ(5,1)A0,A2,A4,A6
    1 FORMAT(4E15.6)
      READ(5,2)U,EL1
    2 FORMAT(2F14.7)
      CB(3)=A2/A0
      WRITE(6,5) A0,A2,A4,A6
    5 FORMAT(4E15.6)
      C(3)=CB(3)
      CB(2)=(A0*A4-A2*A2)/(A0*A0)
      CB(1)=(A0*A0*A6-3.*A0*A2*A4+2.*A2*A2*A2)/(A0*A0*A0)
      CB(1)=CB(1)/CB(2)
      C(2)=CB(2)*5.*(3.-U)/(15.-7.*U)
      C(1)=CB(1)*7.*(15.-7.*U)/(3.*(35.-19.*U))
      D=(1.-C(3))*C(1)+C(2)
      R1=C(1)*C(1)/D
      R1=SQRT(ABS(R1))
      R2=SQRT(ABS(C(2)/D))
      SSI=C(1)/D
      WRITE(6,3)
    3 FORMAT('ORBITAL ELEMENTS:'/)
      WRITE(6,4) R1,R2,SSI
    4 FORMAT('R1=',F14.7,1X,'R2=',F14.7,1X,'SINI**2=',F14.7)
      STOP
      END
END OF FILE
```

```
C PROGRAM TO USE KOPAL'S FREQUENCY DOMAIN METHOD
C TO COMPUTE GEOMETRIC ELEMENTS OF AN ECLIPSING BINARY
C THIS VERSION FOR ANNULAR ECLIPSES - PROGRAM 'ANNULAR'.
C D. HOLMGREN, AUG. '83. MODIFIED SEPT. 18'83
C
      REAL A0,A2,A4,A6,R1,R2,SSI,C(3),CB(3),
     -X,YU,CC(2),YY,INCL,CTI,PI,TAU,DK,RKL
     -,RK,S1,S2,L2,L1,ALF0,Z,PI,XG,AM,FI,RK1,DOC,DTR
      PI=3.14159
C READ IN DATA.
C L1 REFERS TO THE ECLIPSED STAR. DOC IS
C DEPTH OF OCCULTATION, DTR THAT OF THE
C TRANSIT.
      READ(5,3) U,DOC,DTR,L1
    3 FORMAT(4F14.7)
      READ(5,4) A0,A2,A4,A6
    4 FORMAT(4E15.6)
      WRITE(6,5) A0,A2,A4,A6
    5 FORMAT('A0=',E15.6,'A2=',E15.6,'A4=',E15.6,'A5=',E15.6)
      RK=0.5
      RKL=RK
      DO 1 I=1,10
      TAU=2.*(3.*ARSIN(SQRT(RK))-(3.-4.*RK)*(1.+2.*RK)*
     -SQRT(RK*(1.-RK)))/(3.*PI)
      YY=(3.*(1.-U)+2.*U*TAU/(RK*RK))/(3.-U)
      RK=(1.-DTR)/(DOC*YY)
      RK=SQRT(RK)
      DK=ABS(RK-RKL)
      IF(DK.LE.1.E-04)GO TO 2
    1 RKL=RK
    2 CONTINUE
      RK=1./RK
C CONVERT FROM RUSSELL'S K.
C RK IS K, K=R1/R2 (K>1)
      X=(A0*A0*A6-3.*A0*A2*A4+2.*A2*A2*A2)/(A0*A4-A2*A2)
      F2=(15.-7.*U)/(5.*(3.-U))
      F4=3.*(35.-19.*U)/(35.*(3.-U))
      X=X*(A0*(1.+F2*RK*RK)-L1)
      X=X/(A0*A0*(1.+3.*F2*RK*RK+F4*(RK**4.))-3.*A0*L1*(1.+F2*RK*RK)
     -+2.*L1*L1)
      Y=(X*L1-A2)/A0
C X=R2**2 *CSCI**2, Y=COT I**2.
      WRITE(6,18) RK,X,Y,L1,YY
   18 FORMAT('K=',F14.7,'X=',F14.7,'Y=',F14.7,'L1=',F14.7
     -,'YY=',F14.7)
C FIND R1,R2,INCLINATION.
      IF(X .LT. 0.)X=ABS(X)
      IF(Y .LE. 0.)INCL=1.5708
      IF(Y .LE. 0.) GO TO 16
      Y=SQRT(Y)
      INCL=ATAN(1./Y)
   16 WRITE(6,30)
   30 FORMAT('Y<=0. I=90. ASSUMED.')
      R2=X*SIN(INCL)*SIN(INCL)
      R2=SQRT(R2)
      R1=R2*RK
      WRITE(6,21) R1,R2,INCL
   21 FORMAT('R1=',F14.7,'R2=',F14.7,'INCL=',F14.7)
      WRITE(6,22)
   22 FORMAT('THESE ARE FINAL ELEMENTS')
      STOP
      END
END OF FILE
```

```
C PROGRAM TO USE KOPAL'S FREQUENCY DOMAIN METHOD
C TO COMPUTE GEOMETRIC ELEMENTS OF AN ECLIPSING BINARY
C THIS VERSION FOR PARTIAL ECLIPSES - PROGRAM 'PARTIAL'.
C D. HOLMGREN, MAR.4'83.
C
      IMPLICIT REAL(A-H,O-Z)
      REAL INCL,L1
      PI=3.14159
C READ IN DATA.
      READ(5,3) U,DOC,DTR,RK
    3 FORMAT(4F14.7)
      READ(5,4) A0,A2,A4,A6
    4 FORMAT(4E15.6)
      WRITE(6,5) A0,A2,A4,A6
    5 FORMAT('A0=',E15.6,1X,'A2=',E15.6,1X,'A4=',E15.6,1X,'A6=',E15.6)
      DO 100 I=1,2
      RK2=RK*RK
      IF(RK.LT.1.)ALFA=1.-DOC+(1.-DTR)/RK2
      IF(RK.GE.1.)ALFA=1.-DTR+(1.-DOC)*RK2
      L1=A0/ALFA
C RK IS K, K=R1/R2
      X=(A0*A0*A6-3.*A0*A2*A4+2.*A2*A2*A2)/(A0*A4-A2*A2)
      F2=(15.-7.*U)/(5.*(3.-U))
      F4=3.*(35.-19.*U)/(35.*(3.-U))
      X=X*(A0*(1.+F2*RK*RK)-L1)
      X=X/(A0*A0*(1.+3.*F2*RK*RK+F4*(RK**4.))-3.*A0*L1*(1.+F2*RK*RK)
     -+2.*L1*L1)
      Y=(X*L1-A2)/A0
C X=R2**2 *CSCI**2, Y=COT I**2.
      WRITE(6,18) RK,X,Y,L1
   18 FORMAT('K=',F14.7,1X,'X=',F14.7,1X,'Y=',F14.7,1X,'L1=',F14.7)
C FIND R1,R2,INCLINATION.
      IF(X .LT. 0.)X=ABS(X)
      IF(Y .LE. 0.)INCL=1.5708
      IF(Y .LE. 0.) GO TO 16
      Y=SQRT(Y)
      INCL=ATAN(1./Y)
   16 CONTINUE
      R2=X*SIN(INCL)*SIN(INCL)
      R2=SQRT(R2)
      R1=R2*RK
      WRITE(6,21) R1,R2,INCL
   21 FORMAT('R1=',F14.7,'R2=',F14.7,'INCL=',F14.7)
      RK=(A4-A0*Y*Y)/(L1*F2*X*X)+(2.*(Y/X)-1.)/F2
      RK=SQRT(ABS(RK))
  100 CONTINUE
      WRITE(6,22)
   22 FORMAT('THESE ARE FINAL ELEMENTS')
      STOP
      END
END OF FILE
```

```
C PROGRAM TO COMPUTE UNCERTAINTIES IN
C ELEMENTS GIVEN ERRORS OF MOMENTS AND VALUES OF
C ELEMENTS. USES IMSL ROUTINE LEQT1F TO SOLVE
C A SYSTEM OF EQUATIONS.
C BY D. HOLMGREN, FEB.'84.
C
C TO RUN: $R -LOAD#+*IMSLLIB 5=<INPUT FILE> 6=-P.
C
      IMPLICIT REAL(A-H,O-Z)
      REAL INCL
      DIMENSION A(3,3),B(3),WKAREA(10)
C READ IN MOMENTS,ELEMENTS, AND ERRORS.
C SET UP INITIAL ESTIMATE OF DELTA K**2.
      READ(5,1) DA0TR,DA00C,DA2,DA4,DA6,NTYPE
    1 FORMAT(5E15.6,I3)
      READ(5,2)R1,R2,X,Y,INCL,AK,EL1,U
    2 FORMAT(8F14.7)
      READ(5,15) A00C,A0TR,ALF0
   15 FORMAT(3F14.7)
      AK2=AK*AK
      DK2=1.E-03
      IF(NTYPE-1)3,4,3
    3 DALF=DA0TR+AK2*DA00C*(DA00C/A00C+DK2/AK2)
      DL1=(DA00C/A00C-DALF/ALF0)*EL1
      GO TO 5
    4 DALF=DA00C+(DA0TR/AK2)*(DA0TR/A0TR-DK2/AK2)
      DL1=(DA0TR/A0TR-DALF/ALF0)*EL1
    5 CONTINUE
C SET UP EQUATIONS FOR DX,DY,DK**2.
      IF(NYTPE-1)6,7,6
    6 A0=A0TR
      DA0=DA0TR
      GO TO 8
    7 A0=A00C
      DA0=DA00C
    8 CONTINUE
      A(1,1)=EL1
      A(1,2)=-A0
      A(1,3)=0.
      B(1)=DA2+Y*DA0-X*DL1
      F2=(15.-7.*U)/(5.*(3.-U))
      F4=3.*(35.-19.*U)/(35.*(3.-U))
      A(2,1)=2.*EL1*(X-Y+F2*AK2*X)
      A(2,2)=-2.*(EL1*X-A0*Y)
      A(2,3)=EL1*F2*X*X
      B(2)=DA4-Y*Y*DA0
      B(2)=B(2)-(X*X-2.*X*Y+F2*AK2*X*X)*DL1
      XMY=X-Y
      FK2=F2*AK2
      FK4=F4*AK2*AK2
      X2=X*X
      X3=X*X2
      Y2=Y*Y
      XY=X*Y
      A(3,1)=XMY*XMY+FK2*X*(3.*X-2.*Y)+FK4*X2
      A(3,1)=3.*EL1*A(3,1)
      A(3,2)=EL1*(X2-2.*XY+FK2*X2)+A0*Y2
      A(3,2)=-3.*A(3,2)
      A(3,3)=3.*F2*X2*XMY+2.*F4*AK2*X3
      A(3,3)=EL1*A(3,3)
```

```
          B(3)=DA6+Y2*Y*DA0
          Z=X*(X2-3.*XY+3.*Y2)
          Z=Z+3.*FK2*X2*XMY+FK4*X3
          B(3)=B(3)-Z*DL1
          M=1
          N=3
          IA=N
          IDGT=0
          CALL LEQT1F(A,M,N,IA,B,IDGT,WKAREA,IER)
C SOLVE EQUATIONS. X RETURNED IN B.
          WRITE(6,9) (B(I), I=1,3)
        9 FORMAT('DX=',E15.6,1X,'DY=',E15.6,1X,'DK**2=',E15.6/)
          CSI=COS(INCL)*COS(INCL)
          SSI=1.-CSI
          DX=B(1)
          DY=B(2)
          DK2=B(3)
          DCSI=DY*CSI*SSI/Y
          DR2=R2*R2*(DX/X-DCSI/SSI)
          DR1=R1*R1*(DK2/AK2+DR2/(R2*R2))
          WRITE(6,10)DCSI,DR2,DR1
       10 FORMAT('DCOSI**2=',E15.6,1X,'DR2**2=',E15.6,1X,'DR1**2=',E15.6/
C RE-EVALUATE DL1
          IF(NTYPE-1)11,12,11
       11 DALF=DA0TR+AK2*DA00C*(DA00C/A00C+DK2/AK2)
          DL1=(DA00C/A00C-DALF/ALF0)*EL1
          GO TO 13
       12 DALF=DA00C+(DA0TR/AK2)*(DA0TR/A0TR-DK2/AK2)
          DL1=(DA0TR/A0TR-DALF/ALF0)*EL1
       13 CONTINUE
          WRITE(6,14)DL1
       14 FORMAT('NEW DL1=',E15.6)
          STOP
          END
END OF FILE
```

```
      1      0.9020E 00,0.48255E-01,0.37023E-02,0.34278E-03
      2      0.6,0.9092,
END OF FILE
```

Input file, total eclipse (W Delphini).

```
      1      0.6,0.9020,0.7050,0.9020
      2      0.3338,0.018745,0.0015258,0.0001545
END OF FILE
```

Input file, annular eclipse (HS Herculis).

```
      1      0.6,0.9315,0.9462,1.38,
      2      0.538E-01,0.41120E-02,0.62256E-03,0.13489E-03
END OF FILE
```

Input file, partial eclipse (HD219634).

```
     1        0.902000E+00   0.482550E-01   0.370230E-02   0.342780E-03
     2      ORBITAL ELEMENTS:
     3
     4      R1=      0.1533310 R2=      0.2411370 SINI**2=      0.9950880
 END OF FILE
```

Output from program TOTAL.

```
  1      A0=    0.333800E+00A2=   0.187450E-01A4=    0.152580E-02A6=    0.154500E-03
  2      K=      1.8220682X=      0.0278620Y=      0.0191328L1=      0.9020000YY=      1.0857878
  3      Y<=0. I=90. ASSUMED. (IGNORE)
  4      R1=      0.3012697R2=      0.1653449INCL=      1.4333467
  5      THESE ARE FINAL ELEMENTS
END OF FILE
```

Output from program ANNULAR.

```
    1      A0=    0.538000E-01 A2=    0.411200E-02 A4=    0.622560E-03 A6=    0.134890E-03
    2      K=      1.3800001 X=     -0.0147849 Y=     -0.1566744 L1=      0.2919925
    3      R1=      0.1677988R2=      0.1215933INCL=      1.5707998
    4      K=      6.0672550 X=      0.0036558 Y=     -0.0750117 L1=      0.0208900  IGNORE
    5      R1=      0.3668470R2=      0.0604634INCL=      1.5707998
    6      THESE ARE FINAL ELEMENTS
  END OF FILE
```

Output from program PARTIAL.

```
  1      1.7E-03,1.0E-03,2.365E-04,5.599E-05,1.2864E-05,2,
  2      0.301,0.165,0.0278620,0.0191328,1.4333467,1.822,0.9020,0.0980,0.60.,
  3      0.0980,0.3412,1.09,
END OF FILE
```

Program ERROR- input file.

```
  1      DX=  -0.112566E-01 DY=  -0.299121E-01 DK**2=   0.138264E+01
  2
  3      DCOSI**2=  -0.287996E-01 DR2**2=  -0.102001E-01 DR1**2=   0.379044E-02
  4
  5      NEW DL1=   0.662509E-02
END OF FILE
```

Program ERROR- output.

Appendix 4 The WINK8 Program

The light curve synthesis program WINK8 is a modified version of Wood's original WINK program published in 1972 (see Wood (1972)). WINK8 incorporates a number of improvements, only two of which will be mentioned here. The first of these is the use of model stellar atmospheres, since these are more realistic than the blackbody approximation. In the program, these model atmospheres are used as input data and are stored as tables with temperature, surface gravity, and wavelength as the arguments. The tabular quantity is the normal surface flux. These fluxes are used to compute the intensity at each point on the star. An interpolation routine is used to derive the flux, which is then converted to an intensity (assuming the limb darkening to be a known quantity). The second major improvement incorporated in WINK8 is the way in which the reflection effect is computed. Rather than using an explicit numerical integration scheme, WINK8 uses a series of approximations to compute certain quantities required in the calculation of the reflection effect.

WINK8 is not difficult to use. Due to its great size (1857 lines), it is not reproduced here, but a sample data file is shown. The data file is composed of four distinct parts. The first seventeen lines contain input parameters relating to the orbit and to the stars themselves. The first line must contain the word 'WINK' in columns 1-4, and

any other alphabetic data in the remaining columns. Line 2 must be blank. Lines 3-16 contain the orbital and stellar parameters. There are 19 parameters which may be treated as variables, but not all of these should be treated as variables at the same time. In most cases to be encountered, the orbital and stellar parameters to be given specific values are those shown in lines 3-16. The important parameters and their descriptions (along with any necessary comments) may be found in table A4.1.

The "-1 0" must follow the last input parameter. Each input parameter has an index I which appears in columns 1 and 2. If input parameters 1-9 are not known accurately, one need not specify any of them since WINK8 has a set of default parameters. However, such things as the wavelength of observation should be specified. Lines 18-31 contain the model atmosphere data. The format shown is the one which must be used in <u>all</u> cases. Lines 18-24 apply to the primary star, and lines 25-31 to the secondary. The logarithm of the surface gravity appears in lines 18 and 25, and at the end of lines 20, 21, 23, 24, 27, 28, 30 and 31. The temperatures appear in lines 19, 22, 26 and 29. These temperatures are divided into two ranges: low (4000°K - 9500°K) and high (11000°K - 40000°K). These temperatures are followed by two lines of fluxes which correspond to the given temperatures. The wavelengths bracketing the observation wavelength are the second to last numbers on the lines containing the fluxes. The model atmosphere data must be followed by

a blank line. All succeeding lines, except the last two in the file, contain the input data in the form:

phase, magnitude or luminosity, observation weight.

The phase may be in either radians or revolutions. The "-1" in the second last line must be present (anywhere in the first 10 columns). The last line is known as the "T/F card". This line tells WINK8 which of the 19 variable parameters are to be treated as variables and which are to be treated as constants. A "1" denotes that the parameter is to be varied, while a "0" denotes a constant. Note that each number is separated by a space.

To facilitate the construction of a data file, a demonstration file, known as DWINK, may be copied to an empty file and then altered to suit the star being analyzed.

WINK8 will produce several pages of output, so an example is not shown here. However, the output is entirely self-explanatory, so the user should not have any trouble in understanding it.

Table A4.1 Important WINK8 Input Parameters

Parameter	Description	Comments
1	inclination i (degrees)	
2,3,4	e sin ω, e cos ω, T_c	T_c = time of conjunction
5,6	linear limb darkening coefficients u_1, u_2	
7,8	non-linear limb darkening coefficients $U2_1$, $U2_2$	usually set equal to 0.0.
9,10	albedos w_1, w_2	
11	magnitude at quadrature	equal to 0.0 if intensities are used
13,14	temperatures T_1, T_2 (in degrees Kelvin)	one should be held constant
15	unperturbed radius r_1 of primary star	
16	ratio of radii k at primary minimum $\left(k = \frac{\text{radius of eclipsing star}}{\text{radius of eclipsed star}}\right)$	$k < 1$ for an occultation at primary eclipse
19	mass ratio q	$q < 1$ usually
43	wavelength of observation in Å	
45,46	logarithms of the surface gravities, log g_1, log g_2	
84	maximum number of iterations	usually 6
87	equals 0. if data in magnitudes, 1. if intensities are used.	

```
WINK HS HERCULIS.

 1 88.70000
 5 .60000
 6 .60000
11 8.520
13 16000.0
14 10000.0
15 .259000
16 .550000
19 .300000
43 5480.00
45 4.00000
46 4.00000
84 6.00000
87 0.00000
-1 0.
     4.0
   4000.     5500.     6500.     7500.     8500.     9500.
 9.3821E05 2.2382E07 5.2838E07 1.0374E08 1.7648E08 2.4465E08 4950.  4.0
 1.5887E06 2.0354E07 4.0926E07 7.1047E07 1.0566E08 1.3830E08 6000.  4.0
  11000.    13000.    15000.    20000.    30000.    40000.
 3.3779E08 4.6408E08 5.9531E06 9.7285E08 2.1057E09 3.3174E09 4950.  4.0
 1.8437E08 2.4842E08 3.1322E08 4.9357E08 1.0236E09 1.6201E09 6000.  4.0
     4.0
   4000.     5500.     6500.     7500.     8500.     9500.
 9.3821E05 2.2382E07 5.2838E07 1.0374E08 1.7648E08 2.4465E08 4950.  4.0
 1.5887E06 2.0354E07 4.0926E07 7.1047E07 1.0566E08 1.3830E08 6000.  4.0
  11000.    13000.    15000.    20000.    30000.    40000.
 3.3779E08 4.6408E08 5.9531E08 9.7285E08 2.1057E09 3.3174E09 4950.  4.0
 1.8437E08 2.4842E08 3.1322E08 4.9357E08 1.0236E09 1.6201E09 6000.  4.0

 0.0011000 8.9770000 1.0
 0.0014000 8.9880000 1.0
 0.0031000 8.9940000 1.0
 0.0056000 8.9770000 1.0
 0.0120000 8.9710000 1.0
 0.0141000 8.9580000 1.0
 0.0264000 8.8290000 1.0
 0.0285000 8.8090000 1.0
 0.0324000 8.7560000 1.0
 0.0345000 8.7390000 1.0
 0.0366000 8.7150000 1.0
 0.0391000 8.6910000 1.0
 0.0446000 8.6370000 1.0
 0.0548000 8.5620000 1.0
 0.0570000 8.5590000 1.0
 0.0612000 8.5420000 1.0
 0.0756000 8.5360000 1.0
 0.0777000 8.5310000 1.0
 0.0875000 8.5230000 1.0
 0.0984000 8.5340000 1.0
 0.1014000 8.5420000 1.0
 0.1050000 8.5330000 1.0
 0.1084000 8.5380000 1.0
 0.1097000 8.5290000 1.0
 0.1511000 8.5380000 1.0
 0.1532000 8.5330000 1.0
 0.2234000 8.5350000 1.0
 0.2293000 8.5240000 1.0
```

61	0.2327000	8.5280000	1.0
62	0.2382000	8.5270000	1.0
63	0.2404000	8.5190000	1.0
64	0.2433000	8.5400000	1.0
65	0.2539000	8.5220000	1.0
66	0.2565000	8.5190000	1.0
67	0.2620000	8.5030000	1.0
68	0.2658000	8.5210000	1.0
69	0.2755000	8.5360000	1.0
70	0.2734000	8.5370000	1.0
71	0.2755000	8.5110000	1.0
72	0.2779000	8.5200000	1.0
73	0.2804000	8.5250000	1.0
74	0.2846000	8.5120000	1.0
75	0.2868000	8.5290000	1.0
76	0.3111000	8.5090000	1.0
77	0.3158000	8.5020000	1.0
78	0.3177000	8.5100000	1.0
79	0.3220000	8.5150000	1.0
80	0.3279000	8.5230000	1.0
81	0.3430000	8.5140000	1.0
82	0.3485000	8.5090000	1.0
83	0.3536000	8.5230000	1.0
84	0.3674000	8.5050000	1.0
85	0.3696000	8.5090000	1.0
86	0.3780000	8.5100000	1.0
87	0.3805000	8.5050000	1.0
88	0.3835000	8.5000000	1.0
89	0.3916000	8.5050000	1.0
90	0.3937000	8.5030000	1.0
91	0.4136000	8.5100000	1.0
92	0.4162000	8.5060000	1.0
93	0.4204000	8.5030000	1.0
94	0.4259000	8.5050000	1.0
95	0.4285000	8.5070000	1.0
96	0.4306000	8.5090000	1.0
97	0.4357000	8.5240000	1.0
98	0.4383000	8.5280000	1.0
99	0.4408000	8.5320000	1.0
100	0.4467000	8.5530000	1.0
101	0.4497000	8.5620000	1.0
102	0.4505000	8.5690000	1.0
103	0.4530000	8.5730000	1.0
104	0.4572000	8.5860000	1.0
105	0.4696000	8.6240000	1.0
106	0.4722000	8.6720000	1.0
107	0.4769000	8.6140000	1.0
108	0.4785000	8.6120000	1.0
109	0.4797000	8.6250000	1.0
110	0.4823000	8.6230000	1.0
111	0.4844000	8.6250000	1.0
112	0.4866000	8.6220000	1.0
113	0.4895000	8.6250000	1.0
114	0.4916000	8.6170000	1.0
115	0.4963000	8.6220000	1.0
116	0.4984000	8.6190000	1.0
117	0.5007000	8.6320000	1.0
118	0.5049000	8.6260000	1.0
119	0.5071000	8.6240000	1.0
120	0.5113000	8.6280000	1.0

```
0.5147000 8.6180000 1.0
0.5187000 8.5730000 1.0
0.5200000 8.5790000 1.0
0.5239000 8.5720000 1.0
0.5255000 8.5640000 1.0
0.5304000 8.5760000 1.0
0.5354000 8.5580000 1.0
0.5376000 8.5520000 1.0
0.5427000 8.5330000 1.0
0.5456000 8.5270000 1.0
0.5503000 8.5200000 1.0
0.5524000 8.5150000 1.0
0.5546000 8.5170000 1.0
0.5588000 8.5190000 1.0
0.5613000 8.5170000 1.0
0.5638000 8.5200000 1.0
0.5685000 8.5250000 1.0
0.5706000 8.5220000 1.0
0.5728000 8.5210000 1.0
0.5775000 8.5300000 1.0
0.5795000 8.5210000 1.0
0.5842000 8.5090000 1.0
0.6533000 8.5310000 1.0
0.6579000 8.5240000 1.0
0.7011000 8.5220000 1.0
0.7037000 8.5120000 1.0
0.7062000 8.5110000 1.0
0.7104000 8.5260000 1.0
0.7121000 8.5180000 1.0
0.7164000 8.5310000 1.0
0.7189000 8.5260000 1.0
0.8582000 8.5340000 1.0
0.8646000 8.5160000 1.0
0.8756000 8.5340000 1.0
0.8798000 8.5340000 1.0
0.8820000 8.5370000 1.0
0.9053000 8.5490000 1.0
0.9091000 8.5480000 1.0
0.9146000 8.5420000 1.0
0.9172000 8.5430000 1.0
0.9312000 8.5580000 1.0
0.9388000 8.5800000 1.0
0.9456000 8.6070000 1.0
0.9565000 8.6530000 1.0
0.9607000 8.6970000 1.0
0.9646000 8.7440000 1.0
0.9671000 8.7710000 1.0
0.9692000 8.7990000 1.0
0.9752000 8.8700000 1.0
0.9794000 8.9170000 1.0
0.9836000 8.9580000 1.0
0.9862000 8.9550000 1.0
0.9926000 8.9720000 1.0
0.9943000 8.9710000 1.0
0.9964000 8.9700000 1.0
-1.
 1 0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 0 0 0
END OF FILE
```

Appendix 5.

HD219634- observations.

1	0.000,-0.350,1.
2	0.0047000,-0.351,1.
3	0.0073000,-0.354,1.
4	0.0107000,-0.353,1.
5	0.0127000,-0.359,1.
6	0.0148000,-0.363,1.
7	0.0174000,-0.364,1.
8	0.0194000,-0.364,1.
9	0.0229600,-0.369,1.
10	0.0248900,-0.373,1.
11	0.0274300,-0.380,1.
12	0.0299100,-0.381,1.
13	0.0327100,-0.382,1.
14	0.0356400,-0.387,1.
15	0.0377300,-0.392,1.
16	0.0399800,-0.398,1.
17	0.0429500,-0.403,1.
18	0.0448800,-0.404,1.
19	0.0486800,-0.403,1.
20	0.0507700,-0.411,1.
21	0.0560000,-0.412,1.
22	0.0582600,-0.419,1.
23	0.0603000,-0.414,1.
24	0.0643200,-0.416,1.
25	0.0829300,-0.416,1.
26	0.0856900,-0.423,1.
27	0.0881200,-0.418,1.
28	0.0921300,-0.424,1.
29	0.0956000,-0.427,1.
30	0.2751000,-0.416,1.
31	0.2807000,-0.414,1.
32	0.2836000,-0.431,1.
33	0.2864000,-0.422,1.
34	0.2911000,-0.419,1.
35	0.2937000,-0.417,1.
36	0.2963000,-0.417,1.
37	0.3003000,-0.424,1.
38	0.3036000,-0.414,1.
39	0.4305000,-0.410,1.
40	0.4353000,-0.414,1.
41	0.4383000,-0.408,1.
42	0.4478000,-0.400,1.
43	0.4506000,-0.397,1.
44	0.4527000,-0.395,1.
45	0.4548000,-0.392,1.
46	0.4574000,-0.391,1.
47	0.4593000,-0.383,1.
48	0.4617000,-0.386,1.
49	0.4666000,-0.375,1.
50	0.4687000,-0.379,1.
51	0.4721000,-0.374,1.
52	0.4739000,-0.374,1.
53	0.4767000,-0.370,1.
54	0.4788000,-0.369,1.
55	0.4805000,-0.370,1.
56	0.4331000,-0.369,1.
57	0.4927000,-0.365,1.
58	0.4956000,-0.363,1.
59	0.4974000,-0.364,1.
60	0.4996000,-0.363,1.

```
61    0.5017000,-0.361,1.
62    0.5045000,-0.368,1.
63    0.5C69000,-0.369,1.
64    0.5088000,-0.364,1.
65    0.5121000,-0.369,1.
66    0.5156000,-0.367,1.
67    0.5179000,-0.373,1.
68    0.5213000,-0.378,1.
69    0.5324000,-0.396,1.
70    0.5348000,-0.397,1.
71    0.5370000,-0.399,1.
72    0.5400000,-0.403,1.
73    0.5426000,-0.410,1.
74    0.5443000,-0.407,1.
75    0.5461000,-0.409,1.
76    0.5492000,-0.402,1.
77    0.5515000,-0.413,1.
78    0.55360C0,-0.410,1.
79    0.5565000,-0.412,1.
80    0.5600000,-0.414,1.
81.   0.5603000,-0.423,1.
82    0.5774000,-0.416,1.
83    0.5800000,-0.418,1.
84    0.5828000,-0.418,1.
85    0.5848C00,-0.420,1.
86    0.5878000,-0.423,1.
87    0.7716000,-0.432,1.
88    0.7746000,-0.421,1.
89    0.7769000,-0.428,1.
90    0.7796000,-0.428,1.
91    0.7821000,-0.427,1.
92    0.7847000,-0.424,1.
93    0.7869000,-0.421,1.
94    0.7894000,-C.419,1.
95    0.7916000,-0.423,1.
96    0.7937000,-0.425,1.
97    0.7960000,-0.424,1.
98    0.7982000,-0.421,1.
99    0.8032000,-0.420,1.
100   0.8059000,-0.420,1.
101   0.8078000,-0.419,1.
102   C.8107000,-0.425,1.
103   0.8128000,-0.417,1.
104   C.8151000,-0.420,1.
105   0.8181000,-0.421,1.
106   0.8203000,-0.419,1.
107   0.8229000,-0.418,1.
108   0.8247000,-0.417,1.
109   C.8278000,-C.417,1.
110   C.8295000,-0.417,1.
111   0.8339000,-0.420,1.
112   0.8360000,-0.418,1.
113   0.8417000,-0.413,1.
114   0.8438000,-0.419,1.
115   0.9320000,-0.409,1.
116   0.9362000,-0.410,1.
117   0.9379000,-0.409,1.
118   C.9399000,-0.408,1.
119   0.9428000,-0.404,1.
120   0.9450000,-0.401,1.
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121      0.9471000,-0.402,1.
122      0.9501000,-0.397,1.
123      0.9524000,-0.396,1.
124      0.9544000,-0.390,1.
125      0.9567000,-0.388,1.
126      0.9588000,-0.388,1.
127      0.9607000,-0.378,1.
128      0.9633000,-0.375,1.
129      0.9654000,-0.371,1.
130      0.9675000,-0.372,1.
131      0.9703000,-0.366,1.
132      0.9723000,-0.365,1.
133      0.9753000,-0.359,1.
134      0.9790000,-0.357,1.
135      0.9807000,-0.347,1.
136      0.9840000,-0.348,1.
137      0.9862000,-0.348,1.
138      0.9883000,-0.354,1.
139      0.9915000,-0.344,1.
140      0.9935000,-0.349,1.
141      0.9958000,-0.343,1.
142      0.9987000,-0.343,1.
143      1.0000000,-0.350,1.
END OF FILE
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